**MATH 361 Real Analysis** **Homework Assignment 8** **Spring 2024**

**Due by 4pm on April 15. Do not forget to attach the honor code. Each problem is worth 10 points.**

1. Give an example of each of the following, or argue that such a request is impossible.
   (a) A Cauchy sequence that is not monotone.
   (b) A Cauchy sequence with an unbounded subsequence.
   (c) A divergent monotone sequence with a Cauchy subsequence.
   (d) An unbounded sequence containing a subsequence that is Cauchy.

2. Prove the Alternating Series Test by showing that the sequence of partial sums is a Cauchy sequence.

3. Give another proof for the Alternating Series Test using the Nested Interval Property.

4. Prove the Root Test.

5. Determine whether or not the following series converge.
   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k^2}}{100^{100k}}$
   (b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k^4+25k^3}{2k^4+1}\right)^k$
   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{3^k}$
   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{(2k)!}$

   **Hint:** Use the Root Test in (a) and (b), and the Ratio Test in (c) and (d).

6. Use the Cauchy Condensation Test to show that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges if and only if $p > 1$.

7. Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k^4+1}$ converges or diverges.

8. Prove that if the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges, then $\displaystyle\sum_{k=1}^{\infty} |a_k|^2$ converges. Provide a counterexample to show the converse is false.

9. Prove that if $a_k \geq 0$ and $b_k \geq 0$ for all $k$, and both $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ converge, then $\displaystyle\sum_{k=1}^{\infty} a_k b_k$ converges.

10. (a) Prove that if $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\displaystyle\lim_{n\to\infty} a_n = 0$.
    (b) Prove that if $|r| < 1$, then $\displaystyle\lim_{n\to\infty} n^p r^n = 0$ for any $p$.