MATH 361 Real Analysis **Homework Assignment 7** **Spring 2024**

Due by 4pm on April 5. Do not forget to attach the honor code. Each problem is worth 10 points.

1. Prove the sequence $a_n = 5 + (-2/3)^n$ is contractive.

2. Let $b_1 = 2$ and $b_{n+1} = 2 - \frac{1}{5}b_n^2$. Prove that (b_n) is contractive and find its limit. (Hint: First prove that $0 \leq b_n \leq 2$ for all n.)

3. Let $a_1 = 3$ and $a_{n+1} = 20 - \frac{3}{5}a_n$ for $n \geq 1$. Prove that (a_n) is contractive and find its limit.

4. Prove that the sequence
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$$
is convergent, to a limit ≤ 1.

5. Let (a_n) be a sequence in $\mathbb{R}$ and define
$$s_n = a_1 + \ldots + a_n,$$
$$t_n = |a_1| + \ldots + |a_n|,$$
Prove that if (t_n) is convergent, then so is (s_n).

6. Suppose $(a_n) \to a$. Define
$$s_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$
Prove that $(s_n) \to a$.

7. If $A \subset \mathbb{R}$ is nonempty and bounded above, and if $M = \sup(A)$, then there exists a sequence (x_n) in A such that $x_n \to M$.

8. Let $a_n = \dfrac{n!}{n^n}$. Prove that (a_n) converges, and find its limit.

9. Let (a_n) be a sequence in $\mathbb{R}$. Prove that (a_n) is unbounded if and only if there exists a subsequence (a_{n_k}) such that $|a_{n_k}| \geq k$ for all k.

10. True or False (explain):

    (a) If (a_n) is any sequence in $\mathbb{R}$, then the sequence $x_n = \dfrac{a_n}{1+|a_n|}$ has a convergent subsequence.

    (b) If a sequence (a_n) diverges, then every subsequence of (a_n) diverges.

    (c) If a sequence (a_n) diverges, then there exists a subsequence of (a_n) that diverges.

    (d) If there exists a subsequence of (a_n) that diverges, then the sequence (a_n) diverges.