MATH 361 Real Analysis **Homework Assignment 5** **Spring 2024**

Due by 4pm on March 15. Do not forget to attach the honor code. Each problem is worth 10 points.

1. Find the limit of the sequence $x_n = \dfrac{7n-2}{4n+5}$. Use the definition of convergence to prove your assertion.

2. Find the limit of the sequence $y_n = \dfrac{3\sqrt{n}}{\sqrt{n}+5}$. Use the definition of convergence to prove your assertion.

   $\lim y_n = 3$. $|y_n - 3| = \frac{15}{\sqrt{n}+5} < \frac{15}{\sqrt{n}} < \epsilon$, which implies $n > \left(\frac{15}{\epsilon}\right)^2$. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \left(\frac{15}{\epsilon}\right)^2$. Then for all $n \geq N$, we have ...

3. Find the limit of the sequence $z_n = 2 + (3/5)^n$. Use the definition of convergence to prove your assertion.

4. Find the limit of the sequence $x_n = \dfrac{(1+2n)^2}{5+3n+3n^2}$. Use the definition of convergence to prove your assertion.

   $\lim x_n = \frac{4}{3}$. $|x_n - \frac{4}{3}| = \frac{17}{3(3n^2+3n+5)} < \frac{17}{9n} < \epsilon$, which implies $n > \frac{17}{9\epsilon}$. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $N > \frac{17}{9\epsilon}$. Then for all $n \geq N$, we have ...

5. (a) Use the definition of convergence to prove that the sequence $z_n = \sqrt{n}$ does not converge.

   (b) Use the definition of convergence to prove that the sequence $y_n = (-1)^n$ does not converge.

6. Let (a_n) and (b_n) are bounded sequences in $\mathbb{R}$, and $c \in \mathbb{R}$. Prove the following:

   (a) The sequence $(a_n + b_n)$ is bounded.

   (b) The sequence $(a_n - b_n)$ is bounded.

   (c) The sequence $(a_n b_n)$ is bounded.

   (d) The sequence (ca_n) is bounded.

7. Show that the sequence (a_n) defined by

   $$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

   is unbounded.

   Show that $a_{2n} - a_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

8. Show that the sequence (a_n) defined by

   $$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

   is bounded above by 2.

   Prove that $n! \geq 2^{n-1}$ for all n, and use that result and the result in Problem 7.

9. Show that the sequence (a_n) defined by

   $$a_n = \left(1 + \frac{1}{n}\right)^n$$

   is bounded above by 3.

   Use the binomial formula and show that each term in the formula is less than or equal to $\frac{1}{k!}$, where k is the index of the summation.

10. (a) Show that if $\lim x_n = 2$, then there is some $N \in \mathbb{N}$ such that $x_n < 2.03$ for all $n \geq N$.

(b) True or False (explain): If (a_n) and $(a_n b_n)$ are bounded, then (b_n) is also bounded.