MATH 361 Real Analysis **Homework Assignment 2** **Spring 2024**

Due by 4pm on February 9. Do not forget to attach the honor code.

1. (5 points each) Let $a, b, c \in \mathbb{R}$. Prove the following:
   (a) $|ab| = |a||b|$
   (b) $|a+b| \leq |a| + |b|$
   (c) $|a+b| = |a| + |b|$ if and only if $ab \geq 0$.
   (d) If $a < b$ for every $b > c$, then $a \leq c$.

2. (10 points) Prove that for every positive integer n,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

3. (15 points) The **distance** between two points x and y in $\mathbb{R}$ is $d(x,y) = |y - x|$. In class, we showed that $d(x,y)$ satisfies the following:
   (i) $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$ and $d(x,y) = 0$ if and only if $x = y$
   (ii) $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{R}$
   (iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in \mathbb{R}$

   If X is any nonempty set, a function $d : X \times X \mapsto \mathbb{R}$ with the properties (i) - (iii) is called a **metric** (or 'distance function') on X. Verify that the function $D : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined by the formula

$$D(a,b) = \frac{|a-b|}{1 + |a-b|}$$

   is also a metric on $\mathbb{R}$.

4. Let $x, y \in \mathbb{R}$. Use the axioms of the real numbers to prove the following:
   (a) (10 points) If $2 < x < 5$, then $\frac{7}{6} < \frac{2+x}{1+x} < \frac{4}{3}$. (Hint: First rewrite $\frac{2+x}{1+x}$ so that x appears only once.)
   (b) (10 points) $2xy \leq x^2 + y^2$. (Hint: $(x-y)^2 \geq 0$.)

5. (20 points) Let $A = [2, 5)$. Find $\min(A)$, $\max(A)$, $\inf(A)$ and $\sup(A)$. Prove your assertions.

6. (5 points each) Prove that there exists a real number β such that $\beta^3 = 361$, as follows.
   (a) Let $S = \{x \in \mathbb{R} : x^3 < 361\}$. Prove S is nonempty and bounded above, and therefore has a least upper bound β.
   (b) Show that if $\beta^3 < 361$, then there exists $n \in \mathbb{N}$ such that $(\beta + \frac{1}{n})^3 < 361$. Explain why this is a contradiction.
   (c) Show that if $\beta^3 > 361$, then there exists $n \in \mathbb{N}$ such that $(\beta - \frac{1}{n})^3 > 361$. Explain why this is a contradiction.