MATH 361 Real Analysis **Homework Assignment 10** **Spring 2024**

Due by 4pm on May 3. Do not forget to attach the honor code. Each problem is worth 10 points.

1. Let $f : A \to \mathbb{R}$ and let c be a limit point of A. If $\lim_{x \to c} f(x)$ exists, show that it is unique.

2. Prove the Order Limit Theorem for Functional Limits: Let f and g be functions defined on a set A and assume $f(x) \leq g(x)$ for all $x \in A$. Further, let c be a limit point of A and assume that the limits for f and g exist at c. Show that
$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

3. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ and c be a limit point of A. Suppose that $\lim_{x \to c} f(x)$ exists and that $\lim_{x \to c} g(x)$ does not exist.
   (a) If $k \in \mathbb{R}$ is not 0, show that $\lim_{x \to c} kg(x)$ does not exist.
   (b) Show that $\lim_{x \to c} [f(x) + g(x)]$ does not exist.
   (c) Must it be the case that $\lim_{x \to c} [f(x)g(x)]$ does not exist?

4. Use the Sequential Definition of a Functional Limit (Theorem 4 in Note 14) to prove the following limit statements.
   (a) $\lim_{x \to 2} 3x + 1 = 7$
   (b) $\lim_{x \to 2} x^2 = 4$
   (c) $\lim_{x \to 3} 1/x = 1/3$
   (d) $\lim_{x \to 2} x^2 + x - 1 = 5$

5. (a) Give a proper definition in the style of $\epsilon - \delta$ definition for the right-hand and left-hand limit statements:
$$\lim_{x \to c^+} f(x) = L \qquad \text{and} \qquad \lim_{x \to c^-} g(x) = M.$$
   (b) Prove that $\lim_{x \to c} f(x) = L$ if and only if both the right and left-hand limits equal L.

6. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.
   (a) $\lim_{x \to 4} \sqrt{x} = 2,$ where $\epsilon = 1$.
   (b) $\lim_{x \to 3} 5x - 6 = 9,$ where $\epsilon = 1$.

7. Let $f : A \to \mathbb{R}$ and let c be a limit point of A.
   (a) If $\lim_{x \to c} f(x) = L$, show that $\lim_{x \to c} |f(x)| = |L|$.
   (b) Show that it is possible for $\lim_{x \to c} |f(x)|$ to exist even if $\lim_{x \to c} f(x)$ does not exist.

8. Use the divergence criteria to supply a proof for each of the following.
   (a) The function $f(x) = |x|/x$ does not have a limit at $x = 0$.
   (b) The function $f(x) = (x - 1)/(x - 3)$ does not have a limit at $x = 3$.
   (c) The function $f(x) = 1/(x^4 + x^2)$ diverges to infinity at $x = 0$.

9. Let $f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

   Use the Sequential Definition of a Functional Limit to prove that f is continuous at $x = 0$.

10. Let $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases}$.

    Prove that f is discontinuous at $x = 2$.