On Kite Central Configurations

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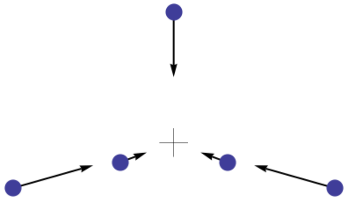
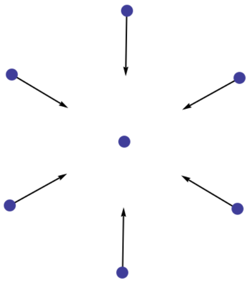
Faculty Research Seminar
College of the Holy Cross
November 29, 2023

Central Configurations

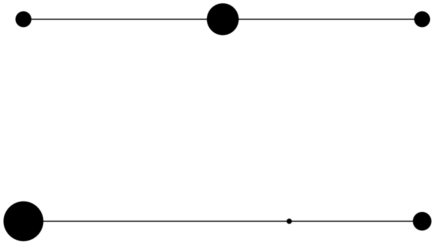




The gravitational force on each body points toward the center of mass and is proportional to the distance from the center of mass.

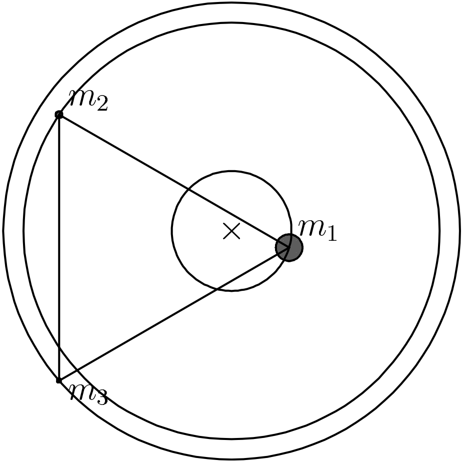
Figures by Rick Moeckel (2014), Scholarpedia, 9(4):10667.

3-Body Collinear Configurations (Euler 1767)



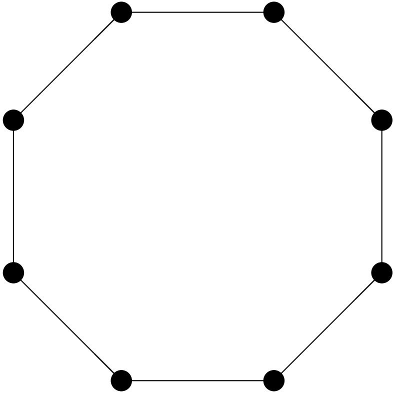
For each ordering of n arbitrary masses on a line, there exists a unique central configuration (Moulton, 1910).

Equilateral Triangle (Lagrange 1772)

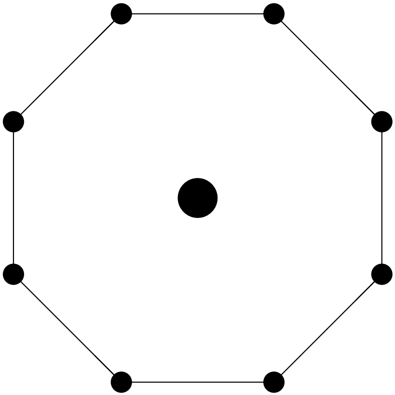


The equilateral triangle is a central configuration for any choice of masses.

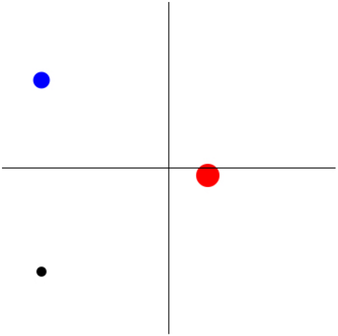
Regular n-gon (equal mass required for $n \geq 4$)



$1 + n$-gon (arbitrary central mass)

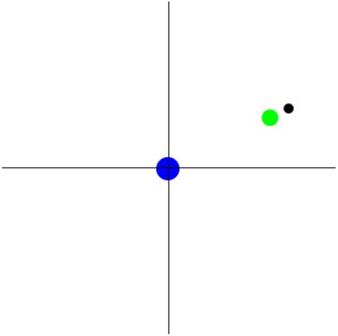


Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize).

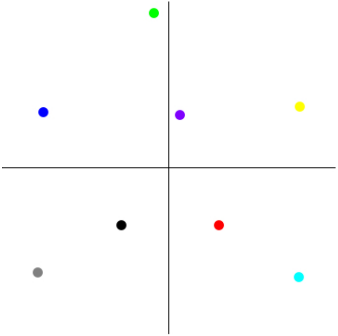


Released from rest, a central configuration maintains the same shape as it heads toward total collision (homothetic motion).

Simulation by Rick Moeckel (2014), Scholarpedia, 9(4):10667.



Given the correct initial velocities, a central configuration will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.



Any Kepler orbit (elliptic, hyperbolic, ejection-collision) can be attached to a central configuration to obtain a solution to the full n-body problem. Above is an example of an asymmetric 8-body c.c. with elliptic homographic motion (eccentricity 0.8).

Simulation by Rick Moeckel (2014), Scholarpedia, 9(4):10667.

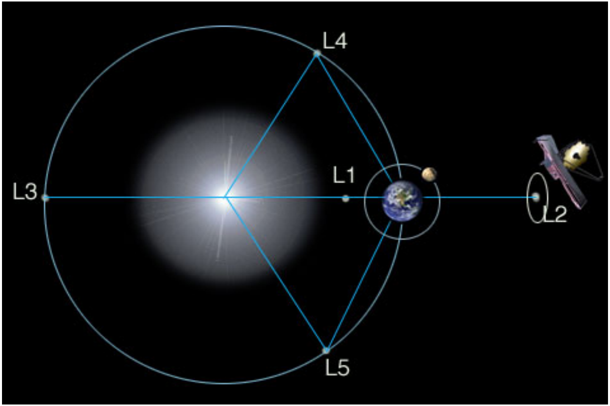


Figure: The five libration points (Lagrange points) in the Sun-Earth system (not drawn to scale). In general, L_4 and L_5 are linearly stable provided the ratio m_{sun}/m_p is sufficiently large. These make great "parking spaces." Many solar observatories (e.g., SOHO) and satellites (e.g., Planck) are located at or around L_1 or L_2.

Source: https://webb.nasa.gov/content/about/orbit.html

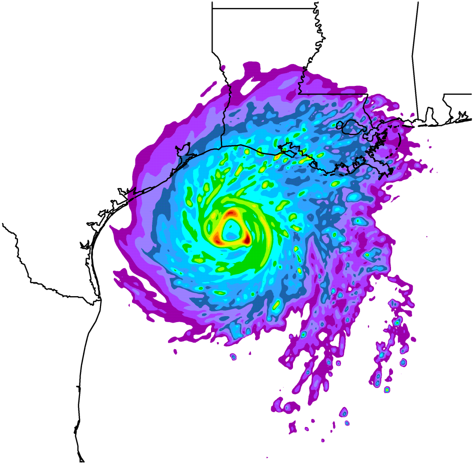


Figure: Weather research and forecasting model from the National Center for Atmospheric Research (NCAR) showing the field of precipitable water for Hurricane Rita (2005). Note the presence of three maxima near the vertices of an equilateral triangle contained within the hurricane's "polygonal" eyewall.
`http://www.atmos.albany.edu/facstaff/kristen/wrf/wrf.html`

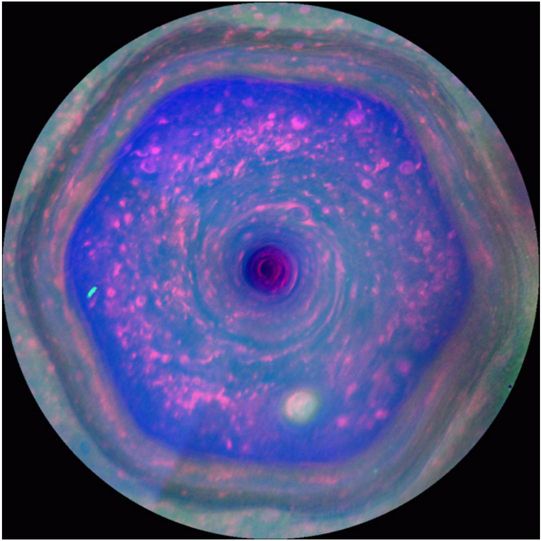


Figure: Saturn's North Pole and its encircling hexagonal cloud structure. First photographed by Voyager in the 1980's and here again recently by the Cassini spacecraft – a remarkably stable structure!

Definition

A planar *central configuration* (c.c.) is a configuration of bodies $x = (x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}^2$ such that the acceleration vector for each body is a common scalar multiple of its position vector (with respect to the center of mass). Specifically, in the Newtonian *n*-body problem with center of mass c, for each index i, $\frac{\partial U}{\partial q_i}(x) = -\lambda m_i(x_i - c)$ or

$$\sum_{j \neq i}^{n} \frac{m_i m_j (x_j - x_i)}{r_{ij}^3} \; + \; \lambda \, m_i(x_i - c) \; = \; 0$$

for some scalar λ independent of i, where $r_{ij} = ||x_j - x_i||$.

- $U(q) = \displaystyle\sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$ is the Newtonian potential function.
- Finding c.c.'s is an algebra problem — no dynamics or derivatives.
- Summing together the n equations above quickly yields $c = \frac{1}{M} \sum m_i x_i$, where $M = \sum m_i$ is the total mass.

Properties of Central Configurations

- Released from rest, a c.c. maintains the same shape as it heads toward total collision (homothetic motion).
- Given the correct initial velocities, a c.c. will rigidly rotate about its center of mass. Such a solution is called a relative equilibrium.
- Any Kepler orbit (elliptic, hyperbolic, ejection-collision) can be attached to a c.c. to obtain a solution to the full n-body problem.
- For any collision orbit in the n-body problem, the colliding bodies asymptotically approach a c.c.
- Bifurcations in the topology of the integral manifolds in the planar problem (holding hc^2 constant where h is the value of the energy and c is the length of the angular momentum vector) occur precisely at values corresponding to central configurations.
- 421 articles found on MathSciNet using a general search for "central configurations" and MSC 70F

Symmetries

Suppose that $x \in \mathbb{R}^{2n}$ is a central configuration. The following are also central configurations:

1. $kx = (kx_1, \ldots, kx_n)$ for any $k > 0$ (scaling; $c \mapsto kc, \lambda \mapsto \lambda/k^3$)
2. $x - s = (x_1 - s, \ldots, x_n - s)$ for any $s \in \mathbb{R}^2$ (translation; $c \mapsto c - s$)
3. $Ax = (Ax_1, \ldots, Ax_n)$ where $A \in SO(2)$ (rotation; $c \mapsto Ac$)

Thus, central configurations are not isolated. It is standard practice to fix a scaling and center of mass c, and then identify solutions that are equivalent under a rotation.

Note: Reflections of x are also central configurations (e.g., multiplying the first coordinate of c and each x_i by -1), but these are regarded as distinct solutions.

An Alternate Characterization of CC's

The system of equations defining a central configuration can be written more compactly as

$$\nabla U(x) + \lambda \nabla I(x) \; = \; 0, \qquad (1)$$

where I is one half the moment of inertia, $I = \frac{1}{2}\sum_{i=1}^{n} m_i ||q_i - c||^2$.

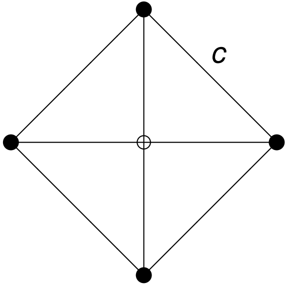
Thus, a central configuration is a critical point of U subject to the constraint $I = k$ (the mass ellipsoid). This gives a useful topological approach to studying central configurations (Smale, Conley, Meyer, McCord, Moeckel, Ferrario, etc.)

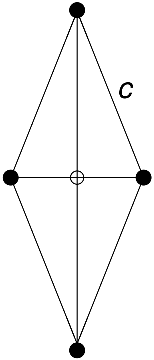
Smale/Wintner/Chazy Question: For a fixed choice of masses, is the number of equivalence classes of planar central configurations finite? (Smale's 6th problem for the 21st century)

Finiteness Question

- $n = 3$ 5 total Euler (1767), Lagrange (1772)
- Collinear $n!/2$ Moulton (1910)
- 4 equal masses 50 total Albouy (1995)
- $n = 4$ Finite, 32 – 8472 Hampton and Moeckel (2006) using BKK Theory
- 5 equal masses 207 total Lee and Santoprete (2009) using the polyhedral homotopy method (HOM4PS-2.0)
- $n = 5$ Albouy and Kaloshin (2012). Finiteness shown for all masses except for a codimension 2 subvariety.
- $n \geq 6$ Open problem!

A Continuum of Central Configurations





Theorem (GR, 1999)

There exists a continuum of c.c.'s in the planar 5-body problem with masses $m_1 = m_2 = m_3 = m_4 = 1$ and $m_5 = -1/4$. The equal masses lie at the vertices of a rhombus and the negative mass is located at the center. The side length remains constant throughout the continuum (triple collision occurs at either ends).

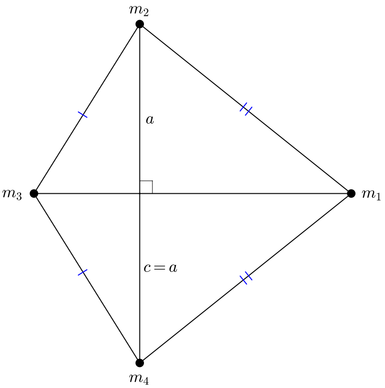
Four-body CC's

- A four-body planar c.c. is called convex if no body lies within the convex hull of the other three. If the configuration is not convex, it is called concave.

- MacMillan and Bartky (1932) proved that for any choice of four masses and for any ordering, there exists a convex c.c. Later, Xia (2004) gave a simpler proof of this fact. Hampton (2002) proved that concave c.c.'s exist for any choice of masses in his doctoral thesis.

- Symmetry: For a convex c.c., if two opposite masses are equal, then the configuration contains a line of symmetry, called a kite configuration (Albouy, Fu, and Sun, 2007). If there are two pairs of equal masses located at two adjacent vertices, then the configuration must be an isosceles trapezoid (Fernandes, Llibre, and Mello, 2017).

Kites



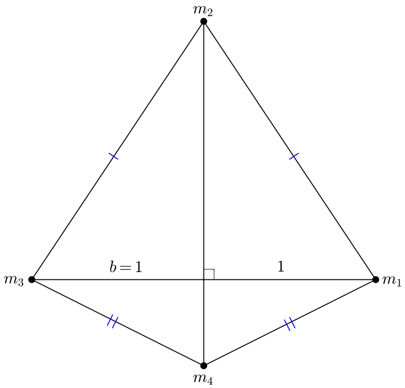


Figure: Two kite central configurations with different symmetry axes. On the left, we must have $m_2 = m_4$, while on the right we have $m_1 = m_3$.

Rhombii

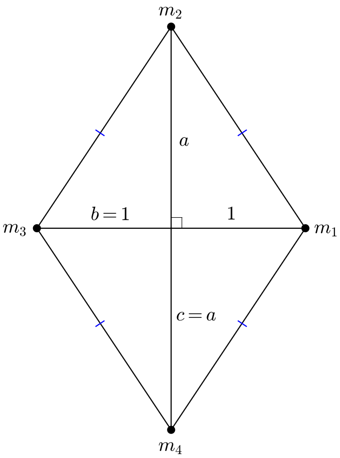
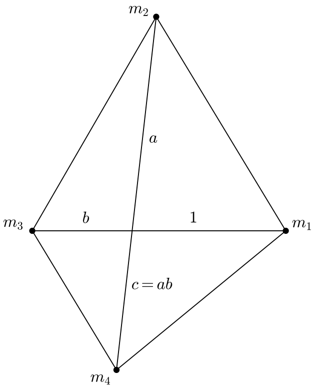


Figure: A rhombus central configuration. The rhombii occur at the intersection of the two types of kites. We must have two pairs of opposite equal masses, $m_1 = m_3$ and $m_2 = m_4$.

Other Examples



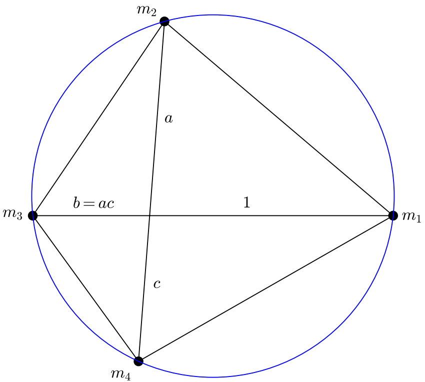


Figure: A trapezoidal c.c. (left) and a co-circular c.c. (right), where the bodies lie on a common circle. Santoprete (2021) has shown that in each case, for a choice of masses where the particular c.c. exists, it is unique (for a fixed ordering of the bodies).

Open problem: Show that there is a unique convex central configuration for any fixed choice of positive masses in a prescribed order. This is Problem #10 on a published list of open problems in celestial mechanics by Albouy, Cabral, and Santos (2012).

- Leandro (2003) proved uniqueness for the convex kite c.c.'s using resultants and the method of rational parametrization.
- Corbera, Cors, and GR (2019) classified the full set of convex c.c.'s (with a focus on symmetric or special geometric configurations) showing the set is three-dimensional.
- Santoprete (2021) proved uniqueness for trapezoidal and co-circular c.c.'s.
- Sun, Xie, and You (2023) gave a numerical computer-assisted proof using interval arithmetic and the Krawczyk Operator for uniqueness on a large subset of masses bounded away from 0.

Research Questions

Goal: Investigate the convex and concave kite c.c.'s for a particular ordering of the bodies, assuming the masses off the line of symmetry are equal.

- Find a "simple," topological argument to prove uniqueness in the convex setting.
- Is it possible to give a geometric construction of a convex kite c.c., and in the process prove uniqueness?
- Why does the argument break down in the concave case?
- Linear stability of the corresponding relative equilibria solutions? How does this depend on the masses and the type of configuration?
- Four-vortex version of the above?

Mutual Distances in the Four-body Problem

Treating the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables, a central configuration is a critical point of

$$U - \lambda(I - I_0) - \frac{\mu}{32} V$$

subject to the constraints $I = I_0$ and $V = 0$, where $I = \frac{1}{2M}\sum_{i<j} m_i m_j r_{ij}^2$, and V is the Cayley-Menger determinant

$$V = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

Key Formula: $\frac{\partial V}{\partial r_{ij}^2} = -32\, A_i A_j$ where A_i is the signed area of the triangle whose vertices contain all bodies except for the ith body.

Dziobek's Equations

$$m_1 m_2(r_{12}^{-3} - \lambda') = \sigma A_1 A_2 \qquad m_3 m_4(r_{34}^{-3} - \lambda') = \sigma A_3 A_4$$

$$m_1 m_3(r_{13}^{-3} - \lambda') = \sigma A_1 A_3 \qquad m_2 m_4(r_{24}^{-3} - \lambda') = \sigma A_2 A_4$$

$$m_1 m_4(r_{14}^{-3} - \lambda') = \sigma A_1 A_4 \qquad m_2 m_3(r_{23}^{-3} - \lambda') = \sigma A_2 A_3$$

where λ' and σ are re-scaled Lagrange multipliers.

This leads to the famous equations of Dziobek (1900):

$$(r_{12}^{-3} - \lambda')(r_{34}^{-3} - \lambda') = (r_{13}^{-3} - \lambda')(r_{24}^{-3} - \lambda') = (r_{14}^{-3} - \lambda')(r_{23}^{-3} - \lambda')$$

Necessary and Sufficient: If these last equations are satisfied for a planar configuration, then the ratios of the masses can be obtained by dividing appropriate pairs in the first list. However, positivity of the masses must still be checked.

Relationships between the mutual distances

In the convex case, two areas are positive and two are negative (e.g., $A_1, A_3 > 0, A_2, A_4 < 0$). Requiring positivity of the masses enforces the following requirements on the mutual distances:

- The diagonals must be longer than all exterior sides.
- The longest and shortest exterior sides are opposite each other.

Simple Consequence: The only possible rectangular c.c. is a square, and the only possible parallelogram is a rhombus.

Assuming that the bodies are ordered sequentially in a counter-clockwise fashion and that r_{12} is the longest exterior side-length, we have the following inequalities:

$$r_{13}, r_{24} \; > \; (\lambda')^{-1/3} \; > \; r_{12} \; \geq \; r_{14}, r_{23} \; \geq \; r_{34}$$

Steps for Proving Uniqueness for Convex Kites

1. Replace the Caley-Menger determinant by a simpler constraint function F that leads to a c.c. when restricted to the set of mutual distances satisfying $r_{12} = r_{14}$ and $r_{23} = r_{34}$.
2. Reduce the dimension of the problem by finding good coordinates for the space. The problem is then recast as finding the critical points of a function on the surface of an ellipsoid.
3. Give a simple topological argument to prove that a critical point must exist.
4. Hard part: Show that any convex critical point must be a minimum. This involves showing that the 2×2 Hessian restricted to the tangent space at a critical point is positive definite.
5. Use basic Morse Theory to conclude that there can only be one such critical point, thereby verifying uniqueness.

Normalized Configuration Space

Assume $m_2 = m_4 = m$ and $M = m_1 + m_3 + 2m = 1$.

Let $r = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \in (\mathbb{R}^+)^6$ and define the normalized configuration space

$$\mathcal{N} = \{r \in (\mathbb{R}^+)^6 : I(r) = 1, F(r) = 0, r_{12} = r_{14}, \text{ and } r_{23} = r_{34}\},$$

where

$$F = r_{13}^2(r_{12}^2 + r_{14}^2 + r_{23}^2 + r_{34}^2 - r_{13}^2 - r_{24}^2) - (r_{12}^2 - r_{23}^2)(r_{14}^2 - r_{34}^2).$$

Key Facts about F:

- F is derived from the geometry of the kite (Pythagorean Theorem).
- F is homogeneous of degree 4.
- $r \in \mathcal{N}$ implies $V(r) = 0$ so the configuration is planar.
- $r \in \mathcal{N}$ implies $\nabla V(r) = 2r_{24}^2 \nabla F(r)$ so the gradients of F and the Cayley-Menger determinant are parallel when restricted to $\mathcal{N}$.

Critical Points on N

$$\mathcal{N} = \{r \in (\mathbb{R}^+)^6 : I(r) = 1, F(r) = 0, r_{12} = r_{14}, \text{ and } r_{23} = r_{34}\}$$

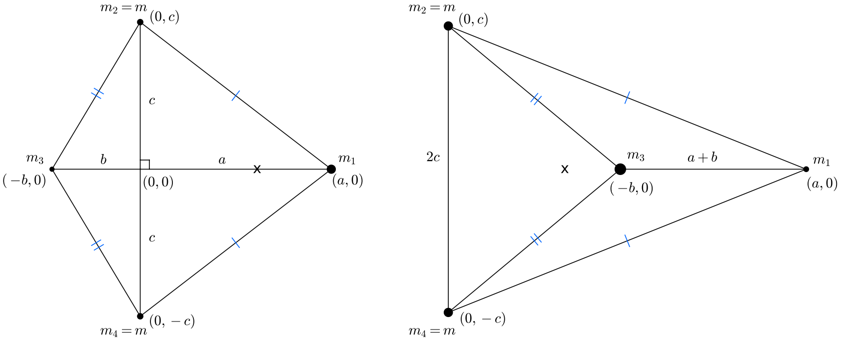
Equality of the two pairs of mutual distances means that $\mathcal{N}$ consists of kite configurations (convex or concave) with m_1 and m_3 on the axis of symmetry and the bodies ordered consecutively in the CC or CCW direction: 1, 2, 3, 4.

Lemma

$r \in \mathcal{N}$ is a kite c.c. with a consecutive ordering iff r is a critical point of U restricted to $\mathcal{N}$.

Key idea: Show that critical points of U on the reduced space $\mathcal{N}$ are also critical points in the full problem, and are therefore central configurations.

Local Coordinates on N



Coordinates: $(a, b, c) \in \mathbb{R}^3$ describe points in $\mathcal{N}$ provided

$$I = (m_1 m + \frac{m_1 m_3}{2})a^2 + (m_3 m + \frac{m_1 m_3}{2})b^2 + m_1 m_3 ab + mc^2 = 1 \, .$$

$ab > 0 \implies$ convex $\qquad ab < 0 \implies$ concave

The configuration space is the ellipsoid

$$\mathcal{C} = \{(a, b, c) \in \mathbb{R}^3 : I(a, b, c) = 1\}.$$

Kite central configurations (convex or concave) are critical points of U restricted to the ellipsoid $\mathcal{C}$.

Different signs for a, b, c lead to different orderings of the bodies (or which body is in the interior for the concave case). Focusing on the convex case, wlog, we assume that $a, b, c \geq 0$.

$$\begin{aligned} r_{12}^2 &= a^2 + c^2 & r_{23}^2 &= b^2 + c^2 \\ r_{13} &= a + b & r_{24} &= 2c \\ r_{14} &= r_{12} & r_{34} &= r_{23} \end{aligned}$$

$$U = \frac{2m_1 m}{r_{12}} + \frac{2m_3 m}{r_{23}} + \frac{m_1 m_3}{r_{13}} + \frac{m^2}{r_{24}}$$

C.C. Equations in New Coordinates

$\nabla U + \lambda \nabla I = 0$ yields

$$
\begin{aligned}
2ma(r_{12}^{-3} - \lambda) &= m_3(a+b)(\lambda - r_{13}^{-3}) \\
2mb(r_{23}^{-3} - \lambda) &= m_1(a+b)(\lambda - r_{13}^{-3}) \\
\lambda &= m_1 r_{12}^{-3} + m_3 r_{23}^{-3} + 2m r_{24}^{-3} \\
r_{12} &= (a^2 + c^2)^{1/2} \\
r_{23} &= (b^2 + c^2)^{1/2} \\
r_{13} &= a + b \\
r_{24} &= 2c \\
I &= 1 \\
m_1 + m_3 + 2m &= 1
\end{aligned}
$$

Symmetry: $a \mapsto b, m_1 \mapsto m_3$ Useful in computations.

Tangent Space

Recall: $I = (m_1 m + \frac{m_1 m_3}{2})a^2 + (m_3 m + \frac{m_1 m_3}{2})b^2 + m_1 m_3 ab + mc^2 = 1$.

We have $\nabla I \; = \; (m_1(a - x), \, m_3(b + x), \, 2mc),$

where $x = m_1 a - m_3 b$ is the first coordinate of the center of mass.

Define the vectors

$$\begin{aligned} v_1 &= [2mc, \, 0, \, -m_1(a - x)]^T \\ v_2 &= [0, \, 2mc, \, -m_3(b + x)]^T . \end{aligned}$$

Since these vectors are linearly independent and satisfy $\nabla I \cdot v_i = 0$, they form a basis for the tangent space to the ellipsoid $\mathcal{C}$ defined by $I = 1$.

Existence of Convex Kite C.C.'s

Consider the space $\mathcal{C}' = \{(a, b, c) \in \mathbb{R}^3 : a, b, c \geq 0, I(a, b, c) = 1\}$.

Theorem (Existence)

Fix a choice of positive masses. The function U restricted to $I = 1$ attains a local minimum in the interior of $\mathcal{C}'$. Consequently, there exists a convex kite central configuration with our prescribed ordering.

Proof outline (modeled after Xia's proof in the general case):

1. Since $\mathcal{C}'$ is compact and U is continuous on its interior, a minimum must exist, although it could be located on the boundary.
2. On the boundary $c = 0$, we have $r_{24} = 0$ and consequently $U \to \infty$, so the minimum is not on this boundary.
3. Pick a point on the boundary $a = 0$ (bodies 2, 1, and 4 are collinear). The directional derivative of U in the direction of the vector $u = v_1/m_1 - v_2/(1 - m_3)$ is negative. Since the vector u points into $\mathcal{C}'$, there cannot be a minimum at this point. A similar argument works for any point on the boundary $b = 0$.

Uniqueness of Convex Kite C.C.'s

Goal: Show that any convex critical point is a nondegenerate local minimum of U restricted to $\mathcal{C}'$.

Let $L = U + \lambda I$. Kite central configurations are critical points of L satisfying $I = 1$.

The Hessian at a critical point $z = (a, b, c)$ is the symmetric matrix $D^2L(z) = D^2U(z) + \lambda D^2I(z)$. We must show that the quadratic form

$$q_z(v) = v^T D^2L(z)v, \quad v \in T_z(\mathcal{C}')$$

is strictly positive for all vectors in $T_z(\mathcal{C}')$.

This is equivalent to showing the matrix

$$P = \begin{bmatrix} v_1^T D^2L(z)v_1 & v_1^T D^2L(z)v_2 \\ v_2^T D^2L(z)v_1 & v_2^T D^2L(z)v_2 \end{bmatrix}$$

is positive definite (two positive eigenvalues), where v_1 and v_2 are the basis vectors for $T_z(\mathcal{C}')$.

Let

$$P = \begin{bmatrix} v_1^T D^2 L(z) v_1 & v_1^T D^2 L(z) v_2 \\ v_2^T D^2 L(z) v_1 & v_2^T D^2 L(z) v_2 \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$$

We want to show $\alpha + \beta > 0$ and $\alpha\beta - \gamma^2 > 0$.

$$\begin{aligned}
\alpha &= 2mm_1c^2[4m^2(\lambda - r_{12}^{-3}) + 2mm_3(\lambda - r_{13}^{-3}) + 6mm_3r_{13}^{-3} \\
&+ 3r_{12}^{-5}(2ma - m_1(a-x))^2 + 3m_1(a-x)^2(m_3r_{23}^{-5} + 8mr_{24}^{-5})] \\
\beta &= 2mm_3c^2[4m^2(\lambda - r_{23}^{-3}) + 2mm_1(\lambda - r_{13}^{-3}) + 6mm_1r_{13}^{-3} \\
&+ 3r_{23}^{-5}(2mb - m_3(b+x))^2 + 3m_3(b+x)^2(m_1r_{12}^{-5} + 8mr_{24}^{-5})] \\
\gamma &= 2mm_1m_3c^2[2m(\lambda + 2r_{13}^{-3}) - 3(a-x)r_{23}^{-5}(2mb - m_3(b+x)) \\
&- 3(b+x)r_{12}^{-5}(2ma - m_1(a-x)) + 24m(a-x)(b+x)r_{24}^{-5})]
\end{aligned}$$

1. Use the first two c.c. equations and $M = 1$ to eliminate the masses:

$$m_1 = m \cdot \frac{2b(r_{23}^{-3} - \lambda)}{(a+b)(\lambda - r_{13}^{-3})}, \quad m_3 = m \cdot \frac{2a(r_{12}^{-3} - \lambda)}{(a+b)(\lambda - r_{13}^{-3})}$$

2. Use the Dziobek equation to eliminate/estimate λ:

$$(r_{12}^{-3} - \lambda)(r_{23}^{-3} - \lambda) = (\lambda - r_{13}^{-3})(\lambda - r_{24}^{-3})$$

3. Scale the three position variables by a factor of $1/c$, so that the new "c" becomes $c = 1$. This does not alter the signs of the eigenvalues. By symmetry, we can assume that $a \geq b$ and $r_{13}, r_{24} > r_{12} \geq r_{23}$.

4. The quantity ab appears in several terms of the trace and determinant. If $ab < 0$, then P is less likely to be positive definite, i.e., a concave kite c.c. is less likely to be a local min.

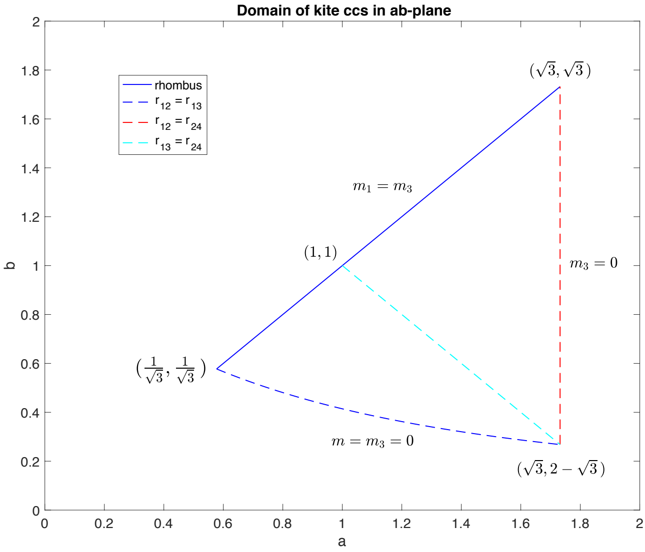


Figure: Region of the *ab*-plane satisfying $r_{13}, r_{24} > r_{12} \geq r_{23}$. Any point outside this region corresponds to a c.c. with a negative mass or $r_{23} < r_{12}$.

Remarks/Future Work:

1. Progress: Can show the trace of P is positive, but the determinant is really complicated!

2. Uniqueness: The Euler characteristic of $\mathcal{C}'$ is 1 (it is homeomorphic to a portion of a solid ellipse). From Morse Theory we have

$$\chi(\mathcal{C}') = 1 = \sum_{i=0}^{2} (-1)^i n_i \,,$$

   where i is the Morse index and n_i is the number of critical points of index i. If all c.p.'s are local mins ($i = 0$), there can be only one.

3. Linear Stability: What region(s) in the ab-plane correspond to stable kites? Do they possess a dominant mass (conjecture of Moeckel's)?

4. Thank you for your attention!