COLLEGE OF THE HOLY CROSS
Department of Mathematics and Computer Science

MATH 361 - Real Analysis
Spring 2024 Final Exam

1	2	3	4	5	6	7	8	9	10	11	12	13	14	total

Your name____

Duration of the Final Exam is 150 minutes. There are 14 problems. The first problem and the second problem are mandatory, and each one of them is worth 10 points. Each of the Problems 3 – 14 is worth 8 points. From Problems 3 – 14, only 10 problems will be graded. If you solve all Problems 3 – 14, you must cross out the two problems in the boxes above that must not be graded. If you solve all Problems 3 – 14 but do not cross out two problems, only the first ten problems from 3 – 14 will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1. Let $A = \left\{ \dfrac{3n+12}{n+2} : n \in \mathbb{N} \right\}$. Find $\min(A)$, $\max(A)$, $\inf(A)$ and $\sup(A)$, and prove your assertions.

2. (a) Find the limit of the sequence $a_n = \dfrac{2n+1}{5n+4}$. Use the definition of convergence to prove your assertion.

   (b) Use the ϵ-δ definition of a functional limit to prove the following limit statement: $\lim_{x \to 1} x^3 + 5 = 6$

   (c) Let

$$f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & x \neq -1; \\ 0 & x = -1 \end{cases}$$

   Prove that f is discontinuous at $x = -1$.

3. (a) Prove that there exists a positive real number β such that $\beta^2 = 7$. Mention any theorem/result used.

   (b) Prove that β is irrational.

4. (a) Define a countable set.

   (b) Prove that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality by constructing a bijection between them.

   (c) Use the Schröeder-Bernstein Theorem to prove $(0,1) \times (0,1) \sim (0,1)$.

5. (a) For any $a \in \mathbb{R}$, prove that $-|a| \leq a \leq |a|$.

   (b) Let (a_n) be a sequence with $(a_n) \to a$.

      i. Prove that $(|a_n|)$ converges to $|a|$.

      ii. Give an example of a sequence (a_n) such that $(|a_n|) \to 1$, but (a_n) does not converge to 1 or -1.

      iii. Prove that if $(|a_n|) \to 0$, then $a_n \to 0$.

6. (a) State the Monotone Convergence Theorem.

   (b) Show that the sequence defined recursively by $a_1 = 3$ and $a_{n+1} = \frac{1}{3}(2a_n + 1)$ converges and find its limit.

   (c) Let (a_n) and (b_n) be Cauchy sequences. Decide whether the sequence $c_n = |a_n - b_n|$ is a Cauchy sequence, justifying your answer.

7. (a) State the Bolzano-Weierstrass Theorem.

   (b) Let $z_1 = \sqrt{2}$ and $z_{n+1} = 6 - 2|z_n - 3|$. Prove that (z_n) has a subsequence that converges.

   (c) Determine whether the following sequences converge. Mention any theorem/result used.

      i. $x_n = \sin\left(\dfrac{n\pi}{2}\right) + \cos n\pi$ ii. $x_n = (-1)^n\left(\pi - \dfrac{1}{n}\right)$ iii. $x_n = 2 + (-1)^n \dfrac{1}{n}$

8. (a) State and prove the Ratio Test.

   (b) Determine whether the following series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10} 10^n}{n!}$

9. Determine whether the following series converges or diverges.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

(d) $\sum_{n=1}^{\infty} \frac{a_n \cos(n\pi)}{n^2}$, where a_n is a sequence converging to $\frac{1}{2}$.

10. (a) Is $\mathbb{I}$ open? Is it closed? Justify your answers.
    (b) Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : f(x) = 0\}$. Show that K is a closed set.
    (c) Does the previous result still hold if we set $K = \{x : f(x) = k\}$, where $k \in \mathbb{R}$ with $k \neq 0$?
    (d) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(r) = 0$ for every $r \in \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

11. (a) Prove that the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x - x^3 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{I} \end{cases}$$

    is continuous at the points $x = 0, \pm 1$, and at no other point.
    (b) Prove that if f is continuous on $\mathbb{R}$, then $|f|$ is also continuous. Is the converse true?

12. (a) Prove that any polynomial function $p : \mathbb{R} \to \mathbb{R}$ is continuous at every point $c \in \mathbb{R}$.
    (b) Prove that any rational function is continuous everywhere on its domain.
    (c) Prove that the function $f(x) = \frac{x-2}{x^2+7}$ attains a maximum and minimum value on the set $[1, 2]$. Mention any result/theorem used.

13. (a) State and prove the Intermediate Value Theorem
    (b) If a and b are positive real numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

    has at least one solution in the interval $(-1, 1)$.

14. Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

    (a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
    (b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.
    (c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$.