Please do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	total
points								

MATH 361 Spring 2024 Exam 2

March 26, 2024

Your name____

The exam has 7 different printed sides of exam problems and 1 side workspace.

Duration of the Midterm Exam is 90 minutes. There are 7 problems. The first problem is mandatory, and it is worth 10 points. Each of the Problems 2 – 7 is worth 8 points. From Problems 2 – 7, only 5 problems will be graded. If you solve all Problems 2 – 7, you must cross out the problem in the box above that must not be graded. If you solve all Problems 2 – 7 and do not cross out a problem, only the first five problems from 2 – 7 will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1. (a) Complete the following definition. A sequence of real numbers (a_n) converges to a real number a, and we write $\lim_{n\to\infty} a_n = a$, if

(b) Find the limit of the sequence $x_n = \dfrac{2n+1}{5n+4}$. Use the definition of convergence to prove your assertion.

(c) Use the definition of convergence to prove that the sequence $z_n = \sqrt{n}$ does not converge.

2. (a) State the Monotone Convergence Theorem.

(b) Use the Monotone Convergence Theorem to prove that the sequence

$$a_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{n+1}\right)$$

is convergent. **Hint:** Calculate a_n for the first few values of n.

3. Let $x_1 = 1$ and $x_{n+1} = 3 + \frac{2}{3}x_n$ for $n \geq 1$.

   (a) Prove (x_n) is monotone.

   (b) Prove (x_n) converges. (You do not need to find its limit.)

4. Let $y_1 = 1$ and $y_{n+1} = 10 - \sqrt{y_n}$.

(a) Prove (y_n) is *not* monotone.

(b) Prove $1 \le y_n \le 9$ for all n.

(c) Show that the sequence (y_n) is contractive.

5. (a) Determine whether the following sequence converges. Mention any theorem used.

$$a_n = \cos\left(\frac{n\pi}{2}\right)$$

(b) Determine whether the following sequence converges. Mention any theorem used.

$$x_n = \frac{1 + (-1)^n}{n}$$

6. (a) Determine whether the following sequence has a subsequence that converges. Mention any theorem used.

$$x_n = \sin\left(\frac{n\pi}{2}\right) + \cos n\pi$$

(b) If (a_n) converges to a and (b_n) converges to b, then prove that $(a_n + b_n)$ converges to $a + b$.

7. (a) Prove that $-1 < \dfrac{e^x - e^{-x}}{e^x + e^{-x}} < 1$ for any $x \in \mathbb{R}$.

(b) Use the Schröeder-Bernstein Theorem to prove $\mathbb{R} \sim (-1, 1)$.

(c) Prove that $\mathbb{I}$ is uncountable. Mention any result used.

WORKSHEET