Please do not write in the boxes immediately below.

problem	1	2	3	4	5	6	total
points							

MATH 361 Spring 2024 Exam 1

February 20, 2024

Your name____

The exam has 7 different printed sides of exam problems and 1 side workspace.

Duration of the Midterm Exam is 90 minutes. There are 6 problems, worth 10 points each. From Problems 1 – 6, only 5 problems will be graded. If you solve all Problems 1 – 6, you must cross out the problem in the box above that must not be graded. If you solve all Problems 1 – 6 and do not cross out a problem, only the first five problems will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1. Let $a, b, c, d \in \mathbb{R}$. Prove the following.

   (a) If $a > 0$, then $a^{-1} > 0$.

   (b) If b and d are nonzero, then $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a \cdot d + c \cdot b}{b \cdot d}$.

(c) $\left|a^{-1}\right| = |a|^{-1}$

(d) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ if $b \neq 0$

2. Let $x_1 = 16$ and $x_{n+1} = 4\sqrt{x_n} - 3$ for all $n \geq 1$. Use induction to prove that $x_n > 9$ for all $n \in \mathbb{N}$.

3. Let A be a nonempty subset of $\mathbb{R}$ and suppose $\beta = \sup(A)$ exists. Decide whether each statement is true or false. You do not need to prove any of your assertions.

   (a) If $x \in A$, then $x \leq \beta$

   (b) If $x \notin A$, then $x > \beta$

   (c) If $x < \beta$, then x is an element of A

   (d) If $x < \beta$, then there exists some $a \in A$ such that $a > x$

   (e) If $x < \beta$, then x is a lower bound of A

   (f) If $x > \beta$, then $x \notin A$

   (g) If $x > \beta$, then x is an upper bound of A

   (h) If $\beta \in A$, then $\beta = \max(A)$

4. (a) Complete the following definition. A number β is the **least upper bound** of a set A, and we write $\beta = \sup(A)$, if

(b) State the Archimedean Property.

(c) Let $A = \{5 - \frac{6}{n} \mid n \in \mathbb{N}\}$. Find $\sup(A)$. Prove your assertion.

5. Let $S = \{x \in \mathbb{R} \,|\, x > 0 \text{ and } x^2 + x < 7\}$.

(a) State the Completeness Axiom.

(b) Prove that $\beta = \sup(A)$ exists.

(c) Prove that if $x \in S$, then there exists $n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$.

(d) Prove $\beta \notin S$.

6. (a) Complete the following definition. Two sets A and B have the same **cardinality** and we write $A \sim B$ if

(b) Define a **countable** set.

(c) Let $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

(i) Prove that $S \sim \mathbb{Q} \times \mathbb{Q}$.

(ii) Is S countable or uncountable? Explain.

WORKSHEET