MATH 133 - 01 **Calculus 1 with Fundamentals** **Worksheet 7**

1. Find the limit.

   (a) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

   (b) $\lim_{x\to 0} \frac{\sin 2x}{2x^2 - x}$

   (c) $\lim_{x\to 0} \frac{\sin 4x}{\sin 7x \cos 4x}$

   (d) $\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$

2. Evaluate the following limit, if it exists. If it does not exist, indicate if it is ∞ or $-\infty$

   (a) $\lim_{x\to \pi^-} x \cot x$

   (b) $\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}$

   (c) $\lim_{x\to 2^+} \frac{x^2 - 2x}{x^2 - 4x + 4}$

   (d) $\lim_{x\to 3} \lfloor x \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function.

3. Let

$$f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ 2 - x^2 & \text{if } 1 < x < 2 \\ x - 3 & \text{if } x \ge 2 \end{cases}$$

   (a) Evaluate the following

   (i) $\lim_{x\to 1} f(x)$ (ii) $\lim_{x\to 2} f(x)$

   (b) Determine where f is continuous expressing your answer in interval notation.

4. (a) Show that

$$f(x) = \begin{cases} x^2 \sin^2(\frac{\pi}{x}) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   is continuous at $x = 0$

   (b) For what value of a is the function continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} ax^2 + 3x & \text{if } x < 2 \\ x^3 - ax & \text{if } x \ge 2 \end{cases}$$

5. Determine all values of the constants A and B so that the following function is continuos for all values of x.

$$f(x) = \begin{cases} Ax - B & \text{if } x \le -1, \\ 2x^2 + 2Ax + B & \text{if } -1 < x \le 1, \\ 4 & \text{if } x > 1, \end{cases}$$

6. (a) Consider the function $f(x) = x + (x - 2 + |x-2|)^2$. Find the limit

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

   if it exists.

   (b) Find $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$, if it exists.

7. Find the vertical asymptotes of $f(x) = \frac{2x+1}{x^2 - 2x - 8}$.

8. Is the following statement true? If it is false, give a counterexample by drawing a graph.

   If $f(x)$ is continuous and has a root in $[a, b]$, then $f(a)$ and $f(b)$ have opposite signs.

9. Assume that $f(t)$ is continuous on $[1, 5]$ and that $f(1) = 20$, $f(5) = 100$. Determine whether each of the following statements is always true, never true, or sometimes true.
   (a) $f(c) = 3$ has a solution with $c \in [1, 5]$.
   (b) $f(c) = 75$ has a solution with $c \in [1, 5]$.
   (c) $f(c) = 50$ has no solution with $c \in [1, 5]$.
   (d) $f(c) = 30$ has exactly one solution with $c \in [1, 5]$.
10. Use the IVT to show that $f(x) = x^3 + x$ takes on the value 9 for some x in $[1, 2]$.
11. Show that $\cos x = x$ has a solution in the interval $[0, 1]$. **Hint:** Show that $f(x) = x - \cos x$ has a zero in $[0, 1]$.
12. Use the IVT to find an interval of length $\frac{1}{2}$ containing a root of $f(x) = x^3 + 2x + 1$.
13. Prove using the IVT.
   (i) $\sqrt{c} + \sqrt{c+2} = 3$ has a solution.
   (ii) $2^x = bx$ has a solution if $b > 2$.
   (iii) $2^x + 3^x = 4^x$ has a solution.
   (iv) $\tan x = x$ has infinitely many solutions.
14. Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$.