How to determine whether a function is increasing/decreasing on an interval?

(a) If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b).

(b) If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).

(1) Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

First Derivative Test Assume that $f(x)$ is differentiable and c is a critical point of $f(x)$. Then we have the following.

(a) If f' changes from $+$ to $-$ at c, then f has a local maximum at c, and $f(c)$ is a local maximum value of f.

(b) If f' changes from $-$ to $+$ at c, then f has a local minimum at c, and $f(c)$ is a local minimum value of f.

(c) If f' is $+$ to the left and right of c, or $-$ to the left and right of c, then f has no local maximum or minimum at c.

Example Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x, \;\; 0 \le x \le 2\pi$$

Definition If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Test for Concavity Assume that $f''(x)$ exists for all $x \in (a, b)$.

(a) If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b).

(b) If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Test for Inflection Points Assume that $f''(x)$ exists. If $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then $f(x)$ has a point of inflection at $x = c$

Example Find the points of inflection and intervals of concavity of $f(x) = 3x^5 - 5x^4 + 1$.

Second Derivative Test Let c be a critical point of $f(x)$. If $f''(c)$ exists, then

- $f''(c) > 0 \implies f(c)$ is a local minimum.
- $f''(c) < 0 \implies f(c)$ is a local maximum.
- $f''(c) = 0 \implies$ inconclusive: $f(c)$ may be a local min, a local max, or neither.

Example Analyze the critical points of

$$f(x) = 2x^3 + 3x^2 - 12x$$

Example Analyze the critical points of

$$f(x) = x^5 - 5x^4$$

Problem Sheet 6

1. Consider the function $f(x) = x^4 - 2x^2 + 3$
   (i) find the intervals on which f is increasing or decreasing.
   (ii) find the local maximum and minimum values of f.
   (iii) find the intervals of concavity and the inflection points.
2. Consider the function $f(x) = x^2 - x - \ln x$
   (i) find the intervals on which f is increasing or decreasing.
   (ii) find the local maximum and minimum values of f.
   (iii) find the intervals of concavity and the inflection points.
3. Consider the function $y = \frac{x^2+1}{(x+3)^2}$
   (i) find all intervals of increase/decrease.
   (ii) find all local extrema.
   (iii) find all intervals of concave up/down.
   (iv) find all points of inflection.
   (v) sketch the graph of the function.
4. Find the local maximum and minimum values of f using both the First and Second Derivative Tests.
   (a) $f(x) = 1 + 3x^2 - 2x^3$
   (b) $f(x) = \dfrac{x^2}{x - 1}$