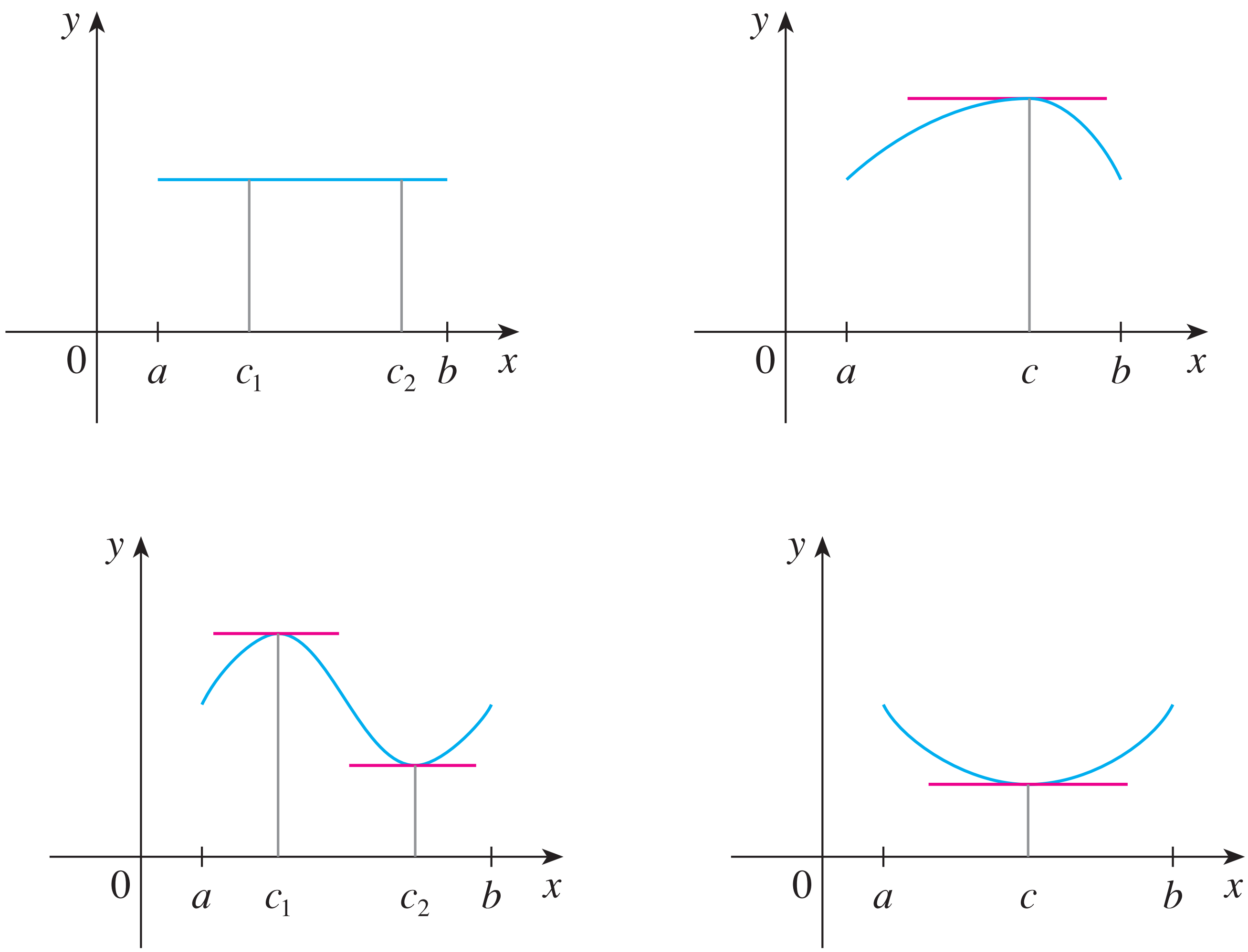
Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



A real-life example of Rolle's Theorem If you throw a ball upward, then its initial displacement is zero ($f(a) = 0$), and when you catch it again, its displacement is zero ($f(b) = 0$). The displacement function $s(t)$ satisfies the conditions of Rolle's Theorem: continuity on $[a, b]$ and differentiability on (a, b). Rolle's Theorem says that there is some instant of time $t = c$ between a and b when $s'(c) = 0$; that is, the velocity is 0, and that will be when the ball reaches its maximum height.

1. Verify Rolle's Theorem for
$$f(x) = x^4 - x^2 \quad \text{on} \quad [-2, 2]$$
2. Show that $f(x) = x^3 + 9x - 4$ has precisely one real root.
3. Determine if Rolle's Theorem can be applied to the function $f(x) = \frac{4x+3}{x^2+1}$ on the interval $[0, \frac{4}{3}]$ and if it can, find all numbers c satisfying the conclusion of the theorem.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

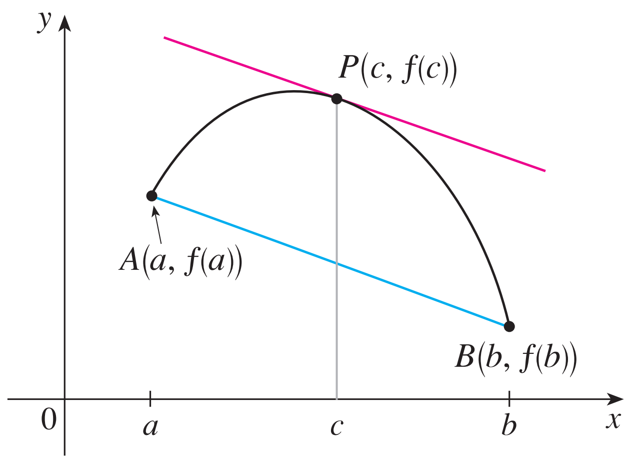
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

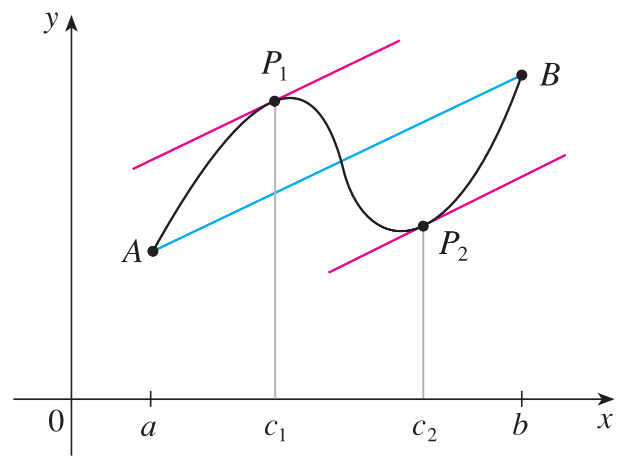
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$





A real-life example of the Mean Value Theorem If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

4. Determine if the Mean Value Theorem can be applied to the function $f(x) = \frac{x-4}{x-3}$ on the interval $[4, 6]$ and if it can, find all numbers c satisfying the conclusion of the theorem.

5. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = 2x^2 - 3x + 1, \quad [0, 2]$$

6. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [Hint: Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

7. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

8. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.