Defintion Prior distributions that result in posterior distributions that are of the same functional form as the prior but with altered parameter values are called *conjugate priors*.

Defintion Let $Y_1, Y_2, \ldots, Y_n$ be a random sample with likelihood function $L(y_1, y_2, \ldots, y_n \,|\, \theta)$, and let θ have prior density $g(\theta)$. The posterior Bayes estimator for $t(\theta)$ is given by

$$\widehat{t(\theta)}_B = E(t(\theta)|Y_1, Y_2, \ldots, Y_n).$$

Example 1 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Bernoulli distribution where $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$ and assume that the prior distribution for p is Beta(α, β). Find the posterior distribution for p.

Example 2 We found the posterior distribution of the Bernoulli parameter p based on a beta prior with parameters (α, β). Find the Bayes estimators for p and $p(1-p)$. [Recall that $p(1-p)$ is the variance of a Bernoulli random variable with parameter p].

Example 3 Suppose that Y is a binomial random variable based on n trials and success probability p. Use the conjugate beta prior with parameters α and β to derive the posterior distribution of $p|y$. Compare this posterior with that found in Example 1.

Example 4 Refer to Example 3.

(1) Suppose that our prior distribution is Beta(5,6), and we observe 10 heads among 30 coin flips. What is our posterior distribution?

(2) Suppose that our prior distribution is Beta(4,16), and we observe 3 heads among 10 coin flips. What is our posterior distribution?

(3) And what if we observe heads on the next coin flip? What does our new posterior look like?

Note: The process could continue for any amount of data! If the initial prior is a beta distribution, then the posterior is always a beta distribution!!

Example 5 In class, we considered an example where the number of responders to a treatment for a virulent disease in a sample of size n had a binomial distribution with parameter p and used a beta prior for p with parameters $\alpha = 1$ and $\beta = 3$.

(1) Find the Bayes estimator for $p =$ the proportion of those with the virulent disease who respond to the therapy.

(2) Derive the mean and variance of the Bayes estimator found in part (a).

Example 6 Refer to Example 3. if Y is a binomial random variable based on n trials and success probability p and p has the conjugate beta prior with parameters $\alpha = 1$ and $\beta = 1$.

(1) Determine the Bayes estimator for p, $\hat{p}_B$.

(2) What is another name for the beta distribution with $\alpha = 1$ and $\beta = 1$?

(3) Find the Mean Square Error (MSE) of the Bayes estimator found in part (a).

(4) For what values of p is the MSE of the Bayes estimator smaller than that of the unbiased estimator $\hat{p} = Y/n$?

So how does this fit in with inference? In the Bayesian context, the parameter θ is a *random variable* with posterior density function $g^*(\theta)$. If we consider the interval (a, b), the posterior probability that the random variable θ is in this interval is

$$P^*(a \leq \theta \leq b) = \int_a^b g^*(\theta)\, d\theta$$

If the posterior probability $P^*(a \leq \theta \leq b) = .90$, we say that (a, b) is a 90% *credible interval* for θ.

Example 7 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from an exponentially distributed population with density $f(y|\theta) = \theta e^{-\theta y}$, $0 < y$. The mean of this population is $\mu = \frac{1}{\theta}$. Use the conjugate gamme (α, β) prior for θ to do the following:

(i) Show that the joint density of $Y_1, Y_2, \ldots, Y_n, \theta$ is

$$f(y_1, y_2, \ldots, y_n, \theta) = \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\,\beta^{\alpha}} \exp\Big[-\theta/\Big(\frac{\beta}{\beta \sum y_i + 1}\Big)\Big].$$

(ii) Show that the marginal density of $Y_1, Y_2, \ldots, Y_n$ is

$$m(y_1, y_2, \ldots, y_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\beta^{\alpha}} \Big(\frac{\beta}{\beta \sum y_i + 1}\Big)^{\alpha+n}.$$

(iii) Show that the posterior density for $\theta|(y_1, y_2, \ldots, y_n)$ is a gamma density with parameters $\alpha^* = n + \alpha$ and $\beta^* = \beta/(\beta \sum y_i + 1)$.

(iv) Show that the Bayes estimator for $\mu = 1/\theta$ is

$$\hat{\mu}_b = \frac{\sum Y_i}{n + \alpha - 1} + \frac{1}{\beta(n + \alpha - 1)}.$$

(v) Show that the Bayes estimator in part (d) can be written as a weighted average of $\overline{Y}$ and the prior mean for $1/\theta$.

(vi) Show that the Bayes estimator in part (d) is a biased but consistent estimator for $\mu = 1/\theta$.

(vii) Assume that an analyst chose $\alpha = 3$ and $\beta = 5$ as appropriate parameter values for the prior and that a sample of size $n = 10$ yielded that $\sum y_i = 1.26$. Construct 90% credible intervals for θ and the mean of the population, $\mu = 1/\theta$.

(viii) Using a conjugate gamma prior for θ with parameters $\alpha = 3$ and $\beta = 5$, we obtained that the posterior density for θ is a gamma density with parameters $\alpha^* = 13$ and $\beta^* = .685$. Conduct the Bayesian test for

$$H_0 : \mu > .12 \quad \text{versus} \quad H_a : \mu \leq .12.$$