Constructing Riemann Surfaces from Puzzle Pieces

S. Allen Broughton
Antonio Costa
Milagros Izquierdo

Rose-Hulman Institute of Technology (emeritus)
Universidad Nacional de Educación a Distancia
Linköpings Universitet

College of the Holy Cross, February 19, 2024

Overview -1

In this talk we will start with a classical motivation for constructing Riemann surfaces from puzzle pieces.

Namely:

- Making complex algebraic functions univalent.
- Integrating rational and algebraic functions in the plane.

After working through simple cases we explore how:

- The puzzle pieces and the (group theory) rulebook to assemble the puzzle may be used to help understand the moduli spaces (complete families) of highly symmetric Riemann surfaces.

Overview - 2

Note:

- The puzzle pieces have an interesting identical shape, (think platonic solids), but we shall not dwell on them much.
- We focus on the *group theory rulebook* used to piece them together.

Needless to say, we will skip lots of detail and focus on special cases that illustrate the main points.

Some of the background work and a forthcoming paper is joint work my research colleagues **Milagros Izquierdo** and **Antonio Costa**.

Integrals from arclength - 1

- In elementary calculus, one of the early integrals we teach students is arclength on the circle

$$\arcsin(x) = \int_0^x \frac{du}{\sqrt{1-u^2}}.$$

- A bit later, when we look at arclength on the ellipse, $v^2/a^2 + u^2 = 1$, we encounter

$$L_a(x) = \int_0^x \frac{\sqrt{1+(a^2-1)u^2}}{\sqrt{1-u^2}} du.$$

  This integral is not elementary and so is typically computed numerically.

Integrals from arclength - 2

- Even later in our curriculum we look at the integrals as functions of a complex variable.

$$\begin{aligned} \arcsin(z) &= \int_0^z \frac{du}{\sqrt{1-u^2}} \\ L_a(z) &= \int_0^z \frac{\sqrt{1+(a^2-1)u^2}}{\sqrt{1-u^2}} du \end{aligned}$$

- But there are issues: the integrand is multi-valued, there are singularities in the integrand, and the value of the integral depends on the homotopy class of the path of integration from 0 to z.

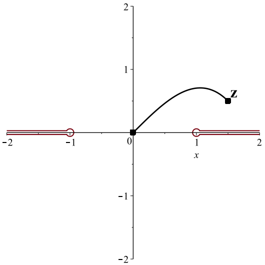
Branch Cuts - 1

- A method to resolve the issues is to use branch cuts so that:
  - the integrand is single valued;
  - we avoid singularities of the integrand; and,
  - we avoid encircling singularities with loops.
- We will pick the branch cuts so that the remainder of the complex plane (cut plane) is simply connected, indeed a curvilinear polygon.
- With this choice, the value of the integral is independent of the path.

Branch Cuts - 2

Here are the branch cuts and an integration path for

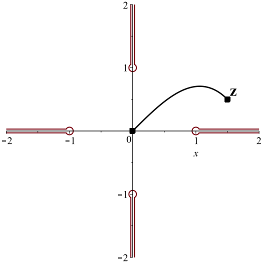
$$\arcsin(z) = \int_0^z \frac{du}{\sqrt{1-u^2}}.$$



Branch Cuts - 3

For $a = \sqrt{2}$, here are the branch cuts and an integration path for arclength of an ellipse:

$$L_{\sqrt{2}}(z) = \int_0^z \frac{\sqrt{1+u^2}}{\sqrt{1-u^2}}du.$$



Abelian integrals - 1

The preceding discussion can be extended to Abelian integrals of algebraic functions that were studied by Abel and others. Given a rational function $R(u, v)$, we consider integrals of the form

$$I(z) = \int_{z_0}^{z} R(u, v)du$$

such that:

- v is a locally defined function that satisfies

  $$F(u, v) = r_0(u)v^n + r_1(u)v^{n-1} + \ldots + r_n(u) = 0$$

  for some rational functions $r_i(u)$; and,
- the path of integration

  $$\gamma(t), 0 \leq t \leq 1, \gamma(0) = z_0, \gamma(1) = z$$

  does not pass through any singularities of the integrand.

Abelian integrals - 2

Here are several classes of such integrals:

- if
$$F(u,v) = v^2 - f(u)$$
where $f(u)$ is a polynomial of degree 3 or 4, then $I(z)$ is called *elliptic*;
- if
$$F(u,v) = v^2 - f(u)$$
where $f(u)$ is a rational function of "higher degree", then $I(z)$ is called *hyper-elliptic*; and,
- if p is a prime
$$F(u,v) = v^p - f(u)$$
where $f(u)$, is a rational function then $I(z)$ is called *p-gonal* or *super-elliptic*.

Transition to a Riemann surface - 1

We will be guided by the super-elliptic case:

- $$F(u,v) = v^p - f(u) = v^p - \prod_{j=1}^{r}(u-a_j)^{t_j}, \qquad (2)$$
  where a_j, t_j, and $t = t_1 + \cdots + t_r = \deg(f)$ satisfy these conditions
  - the a_j are distinct complex numbers;
  - $0 < t_j < p$; and
  - p divides t.
- The a_i are called branch points.
- $v = \sqrt[p]{f(u)}$ is a multi-valued, locally holomorphic function, except at the branch points.
- Any two local solutions v_1, v_2 of (2) are related by multiplying by a p'th root of unity, ω: $v_2(u) = \omega v_1(u)$, valid in a small disc not meeting the branch points.

Transition to a Riemann surface - 2

Show Ball (next slide)
Make a *cut system* with branch cuts that satisfy:

- the cuts are non-intersecting paths that connect branch points to infinity,
- the remainder of the complex plane (cut plane) is simply connected, in fact is an open curvilinear polygon.

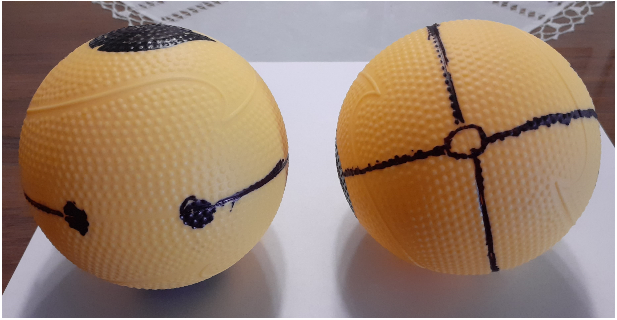
Now, in our Abelian integral we have

- the integrand is single valued though discontinuous along the branch cuts;
- we avoid singularities in the integrand; and,
- we avoid encircling any collection of branch points with a loop.

The function v and, hence, the integrand can be analytically continued across a branch cut, away from singularities.

Show Ball

Two views of the Riemann sphere with 4 cone points and a cut system based at infinity. Left view at 0 and right view at infinity. Black dots or blobs are cone points, lines are cuts and small open circle is infinity.



Transition to a Riemann surface - 3

We now have:

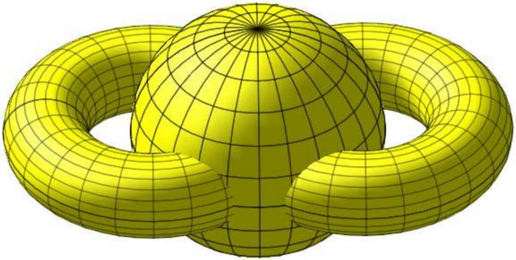
- p copies of the cut plane with the same branch cuts (these are the puzzle pieces);
- p different integrals with different integrands related by

$$R_j(u, v_j) = R_i(u, \omega_{i,j} v_i)$$

  for a suitable root of unity $\omega_{i,j}$; and,
- for an arbitrary path that crosses the branch cuts, we have to work with each piece individually in each of the cut planes.
- In the preceding bullets the $\omega_{i,j}$ play the role of the group theoretic rulebook for assembling the Riemann surface.

Transition to a Riemann surface - 4

- Another approach to Abelian integrals is to work with the closed Riemann surface S defined by F.
- S is a compact, one dimensional complex manifold.
- S is sphere with σ handles attached ($\sigma = 2$ pictured).
- Surfaces of the same genus σ are homeomorphic.
- Surfaces are *conformally equivalent* if there is a biholomorphic map $h : S_1 \to S_2$.



Transition to a Riemann surface - 5

Here is an outline of the "complex algebraic curve" construction of the Riemann surface of an algebraic function defined by $F(u, v) = 0$:

- define
$$S^\circ \subset \mathbb{C} \times \mathbb{C} = \{(u, v) : F(u, v) = 0\}\,;$$
- compactify S° by adding points at infinity; and,
- resolve any singularities (non-manifold or cone points).

Transition to a Riemann surface - 6

Alternatively, here is a topological construction using branch cuts and gluing (puzzle pieces):

- Define the branch set, B, of F as the set of $a \in \mathbb{C}$ such that

$$r_0(a)v^n + r_1(a)v^{n-1} + \ldots + r_n(a) = 0$$

  has fewer than n roots. In super-elliptic case $f(u) = 0$, or $a = a_j$. See (2).
- Using the branch set, construct n different cut planes, each with a different branch of v defined on the cut plane.
- Using analytic continuation, glue the cut planes along the branch cuts so that v is continuous. Note that different sides of a branch cut may get glued to different cut planes.
- Compactify the resulting surface by adding points at infinity and any points corresponding to branch points.
- Resolve any singularities.

Abelian integrals on a Riemann surface - 1

Here is where we stand.

- We constructed a Riemann surface S with two important meromorphic functions. Using the algebraic curve construction we define:

$$\begin{aligned} u &: S \to \widehat{\mathbb{C}},\ (u,v) \to u, \\ v &: S \to \widehat{\mathbb{C}},\ (u,v) \to v. \end{aligned}$$

- The branch cut construction dissects our surface into n polygons. The boundary arcs of the polygon connect points lying over infinity to points lying over the branch points. The function u lifts the base cut plane biholomorphically to the polygons.

Abelian integrals on a Riemann surface - 2

- The field of meromorphic functions on S, $\mathbb{C}(S) = \mathbb{C}(u, v)$, is a degree n extension of the rational function field $\mathbb{C}(u)$.
- $R(u, v)$ is a meromorphic function on S and, so, the Abelian integral

$$\int_{z_0}^{z} R(u, v)du$$

is the same as the integral of the meromorphic differential form $R(u, v)du$ on a path lifted to S.
- If $\mathbb{C}(u, v)$ is a *Galois extension* of $\mathbb{C}(u)$, then S has a group G of n automorphisms such that $S/G = \widehat{\mathbb{C}}$.
- In the super-elliptic case the group $G = \{\omega \in \mathbb{C} : \omega^p = 1\}$
- We will focus on the Galois case.

The three geometries

Another approach to Riemann surfaces is through geometry and group theory. This is an effective approach for:

- constructing surfaces; and
- working with automorphisms of surfaces.

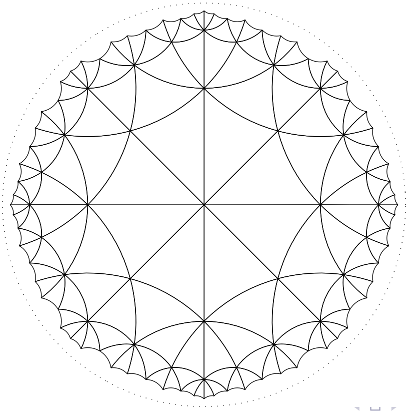
There are three geometries to consider:

- spherical - genus 0 - based on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{S}^2$;
- euclidian - genus 1 - based on the complex plane $\mathbb{C}$; and,
- hyperbolic - genus 2 or greater - based on the hyperbolic plane $\mathbb{H}$.
- We will focus on hyperbolic geometry.

Hyperbolic geometry

Explain the disc model of $\mathbb{H}$ via the picture, which is a tiling of the hyperbolic plane by congruent triangles.

4−4−3 tiling



(3)

Automorphisms of hyperbolic geometry

Using the Poincaré disc model for hyperbolic geometry, the conformal isometries are:

- hyperbolic - translation with axis - two fixed points on the boundary;
- elliptic - rotation (typically of finite order) - one fixed point in the disc the other outside the disc; and
- parabolic - one fixed point on the boundary of the disc - we don't work with these in this talk.
- Every conformal automorphism is a linear fractional transformation and has the form

$$T(z) = \frac{az + b}{\overline{b}z + \overline{a}}.$$

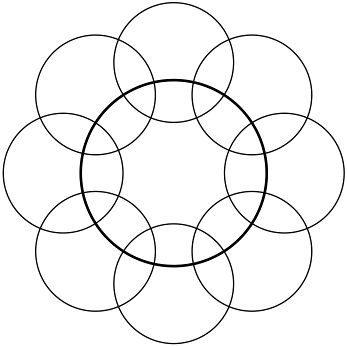
Higher genus surfaces as a quotient - 1

For a surface S of genus $\sigma \geq 2$, we have the following:

- a convex fundamental polygon $P \subset \mathbb{H}$ with 4σ sides;
- 2σ side pairing transformations $T_1, \ldots, T_{2\sigma}$ of P;
- the group $\Pi = \langle T_1, \ldots, T_{2\sigma} \rangle$ acts without fixed points on $\mathbb{H}$;
- the Π-transforms of P tile $\mathbb{H}$; and,
- the quotient space $\mathbb{H}/\Pi$ is conformally equivalent to S.
- The group Π is called a *Fuchsian surface group*.

Higher genus surfaces as a quotient - 2

The central octagon is a fundamental polygon for a genus 2 surface.



Automorphism groups of surfaces

An automorphism of a Riemann surface is a conformal homeomorphism of the surface.

- The automorphism group $\mathrm{Aut}(S)$ is finite and

$$|\,\mathrm{Aut}(S)| \leq 84(\sigma - 1).$$

- For a super-elliptic surface, $\mathrm{Aut}(S)$ contains a cyclic group of order p in this format:

$$(u, v) \to (u, \omega v),\ \omega^p = 1,$$

since

$$F(u, \omega v) = (\omega v)^p - f(u) = v^p - f(u) = F(u, v).$$

- Otherwise, automorphisms of a generic Riemann surface are hard to find.

Start with the automorphism group! - 1

To find any surface with automorphisms, we do the following:

- find a Fuchsian group Γ with a nice generating system;
- determine a suitable, finite index, normal, surface group (no elliptic elements)
$$\Pi \lhd \Gamma;$$
- Then, the finite group Γ/Π acts naturally as a group of automorphisms on the surface $S \simeq \mathbb{H}/\Pi$.
- We are going to assume that $\mathbb{H}/\Gamma$ is a sphere with cone points.

Start with the automorphism group! - 2

Here is a non-generic way to construct a Γ and a Π.

- Pick an r-sided convex polygon P in $\mathbb{H}$ such that successive counter-clockwise edges e_i, e_{i+1} meet an angle of $\frac{\pi}{m_i}$. See Figure 3.
- Let $\gamma_i = r_i r_{i+1}$ be the product of the reflections in e_i and e_{i+1}. The element γ_i is an elliptic rotation of order m_i fixing the vertex $e_i \cap e_{i+1}$.
- Let

$$\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle .$$

  a group generated by rotations.
- P is half a fundamental polygon for Γ.

Start with the automorphism group! - 3

- Γ has this presentation:

$$\Gamma = \left\langle \gamma_1, \ldots, \gamma_r : \prod_{j=1}^{r} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \right\rangle$$

Continuing:

- Pick an epimorphism $\eta : \Gamma \to G$ that *preserves the order of elliptic elements*. This is called a *surface kernel epimorphism* or a *smooth epimorphism*.
- $\Pi = \ker(\eta)$ is the desired surface group, and $G \simeq \Gamma/\Pi$ acts as a group of automorphisms of $S \simeq \mathbb{H}/\Pi$. The inverse map $\epsilon : G \hookrightarrow \mathrm{Aut}(S)$ of $\overline{\eta} : \Gamma/\Pi \leftrightarrow G$ is an *action* of G on S determined by the surface kernel epimorphism.

Start with the automorphism group! - 4

- Let $c_i = \eta(\gamma_i)$. The r-tuple $(c_1, \ldots, c_r)$ satisfies:

$$\begin{aligned} G &= \langle c_1, \ldots, c_r \rangle\,, \\ o(c_i) &= m_i, \\ c_1 \cdots c_r &= 1. \end{aligned}$$

  Such an r-tuple is called a generating $(m_1, \ldots, m_r)$-vector of G. The c_i are rotations on S of order m_i with one or more fixed points.
- The r-tuple $(m_1, \ldots, m_r)$ is called the signature of the G-action.
- Surface kernel epimorphisms $\eta : \Gamma \to G$ are in 1-1 correspondence with the finitely many generating vectors.

Start with the automorphism group! - 5

The surfaces S have genus

$$\sigma = 1 + \frac{|G|}{2}\left(r - 2 - \sum_{j=1}^{r} \frac{1}{m_j}\right)$$

Moduli of integrals and surfaces

We previously promised to link the foregoing to moduli space.

- On an ellipse the integral of arclength depends on a parameter a.

$$L_a(x) = \int_0^x \frac{\sqrt{1+(a^2-1)u^2}}{\sqrt{1-u^2}}\,du$$

- The corresponding Riemann surfaces S_a also depends on the parameter a.
- The total arclength of the ellipse is (see Fig (1), use Maple)

$$2\int_{-1}^{1} \frac{\sqrt{1+(a^2-1)u^2}}{\sqrt{1-u^2}}\,du = 4 EllipticE\left(\sqrt{a^2-1}\right)$$

- The arclength corresponds to an Abelian integral over a closed loop that exists in all the surfaces S_a, and so is a *measure of the conformal type* of S_a.

Moduli space

- For genus $\sigma > 1$ a typical surface is “determined” by $3\sigma - 3$ complex parameters or moduli.
- There are many candidates for the moduli.
- The moduli space $\mathcal{M}_\sigma$ is the set of all conformal equivalence classes of surfaces of genus σ.
- The moduli can be used to coordinatize $\mathcal{M}_\sigma$.
- $\mathcal{M}_\sigma$ is a complex algebraic variety of dimension $3\sigma - 3$, with a complicated singularity set $\mathcal{B}_\sigma$ (discussed next frame).

Strata in the branch locus - 1

- For large enough σ, the singularity set $\mathcal{B}_\sigma$ consists of those surfaces with non-trivial (or exceptional) automorphisms and is called the *branch locus*.
- The branch locus consists of various *equisymmetric strata* $\mathcal{S}$ corresponding to the finitely many possible automorphism groups.
- Each stratum corresponds to the family of surfaces whose automorphism groups are "the same" (need mapping class group to define).
- Considerable work on the branch locus (especially super-elliptic curves) has been done by M. Izquierdo, A. Costa, G. Bartolini, H. Parlier, as well as others.

Strata in the branch locus - 2

- Sometimes we can construct families of surfaces with automorphisms directly.
- Families of super-elliptic surfaces are pervasive, with equation and genus,

$$v^p = \prod_{i=1}^{r} (u - a_i)^{t_i}, \sigma = \frac{(p-1)(r-2)}{2}.$$

- These families determine strata of dimension $r - 3$. Of special interest is the hyperelliptic locus with $p = 2$.
- The numerous images of the super-elliptic families form major pieces of the branch locus.

Strata in the branch locus - 3

- Consider surfaces S with a G action with signature $\mathfrak{s} = (m_1, \ldots, m_r)$.
- The finitely many *group action strata* $\mathcal{S}$ of these surfaces can be derived from the components of a branched covering
$$q : \mathcal{S}_{G,\mathfrak{s}} \to \mathcal{B}_{G,\mathfrak{s}}$$
where $\mathcal{B}_{G,\mathfrak{s}}$ is a nice open set in $\widehat{\mathbb{C}}^{r-3}$ (complement of some hyperplanes).
- the "horizontal" classifier (continuous moduli, image of q) is the quotient surface $S/G = \mathbb{H}/\Gamma = \widehat{\mathbb{C}}$ (with r cone points).
- the " vertical" classifier (discrete moduli, finite fibres of q) is the $\mathrm{Aut}(G)$ class of the smooth epimorphism $\eta : \Gamma \to G$.

Tilings from puzzle pieces - setup

From a geometric perspective, a

$$\text{“smooth epimorphism } \eta : \Gamma \to G\text{”}$$

is a somewhat unsatisfying classifier, so we use a puzzle piece construction.

To recap we have:

- A quotient map
$$\pi_G : S \to S/G = \widehat{\mathbb{C}}.$$
- The map is branched over the cone points $z_1, \ldots, z_r \in \widehat{\mathbb{C}}$.
- Over the cone point z_j, the quotient map has the local form $z \to z^{m_j}$.

Tilings from puzzle pieces - cut systems -1

Select a suitable cut system (embedded graph, see next slide) $\mathcal{E} \subset \widehat{\mathbb{C}}$ such that:

- The vertices of $\mathcal{E}$ contain all the cone points and perhaps one regular point.
- The arcs of $\mathcal{E}$ are smooth, meet only at vertices, and have definite, distinct tangents at the vertices.
- The complement $\widehat{\mathbb{C}} - \mathcal{E}$ is an open polygon.
- diagrams of the some sample cut systems and the corresponding polygon models are shown on the next two slides.

Tilings from puzzle pieces - cut systems - 2

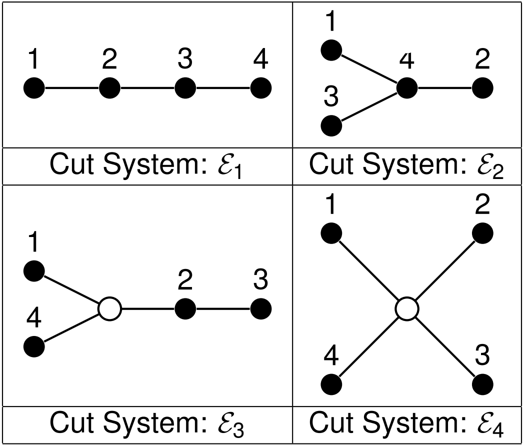


Figure 1: Various cut systems $\mathcal{E}_i$

Tilings from puzzle pieces - polygons - 1

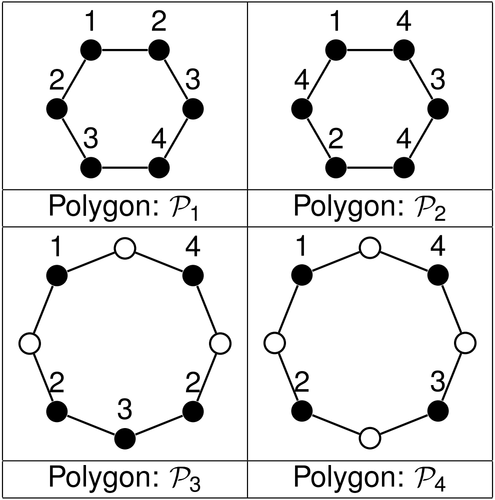


Figure 2: Various model polygons $\mathcal{P}_i$

Tilings from puzzle pieces - polygons - 2

Using the branched cover π_G, lift up the cut system $\mathcal{E}$ to an embedded graph $\tilde{\mathcal{E}}$ on S.

Polygons (puzzle pieces):

- $S - \tilde{\mathcal{E}}$ is a disjoint union of open polygons, each of which is conformally equivalent to $\widehat{\mathbb{C}} - \mathcal{E}$, and upon which G acts simply transitively.
- Select a distinguished polygon $\mathcal{P}^\circ$, and label the open polygons via
$$g \leftrightarrow g\mathcal{P}^\circ.$$

Edges:

- In $\mathcal{E}$, set $k = \#arcs = \#nodes - 1$. In S, the open polygon $\mathcal{P}^\circ$, has a boundary of $2k$ edges, $e_1, \ldots, e_{2k}$ in counter-clockwise order.
- Each edge e_i has an oppositely oriented edge $e_j = e_i^{op}$

Tilings from puzzle pieces - polygons - 3

Edges (continued):

- Each open polygon $g\mathcal{P}^\circ$ in $S - \tilde{\mathcal{E}}$ has a boundary of oriented edges of the same type as $\partial\mathcal{P}^\circ$, via

$$e_i \leftrightarrow ge_i.$$

Vertices:

- Over a white node angles are preserved.
- Over the node z_j the angles are divided by m_j (look at example).

Tilings from puzzle pieces - Rulebook

- The open polygon $\mathcal{P}^\circ$ meets a unique polygon $\tau_{e_i}\mathcal{P}^\circ$ along the edge e_i of $\mathcal{P}^\circ$.
- **Rulebook:** The following is easily proven. For $g \in G$

$$g\mathcal{P}^\circ \text{ meets } (g\tau_{e_i})\mathcal{P}^\circ \text{ along } ge_i \subset \partial g\mathcal{P}^\circ \qquad (4)$$

- τ_{e_i} is called a crossover transform, and

$$\tau_{e_i^{op}} = \tau_{e_i}^{-1}$$

- The crossover transformations are easily computed from the generating vector.

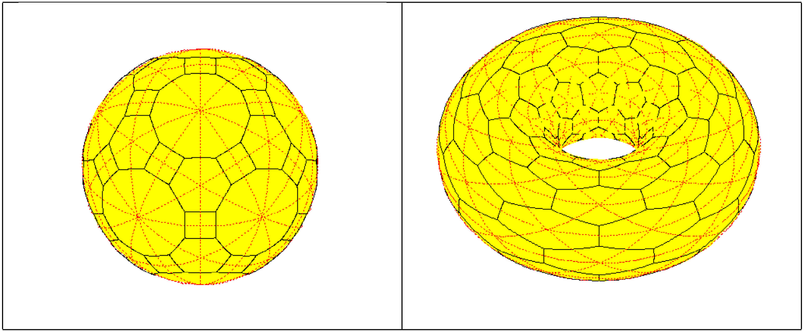
Cayley Graphs - 1

Embedded Cayley graph (geometrical): (See figures on next slide)

- Vertices: Pick $v_0 \in \mathcal{P}^\circ$. The vertices are $\{gv_0 : g \in G\}$.
- Edges: For each e_i and $g \in G$, pick an arc in S from gv_0 to $(g\tau e_i)v_0$, that crosses ge_i transversely.
- The first portion lies in $g\mathcal{P}^\circ$, crosses ge_i transversally in a single interior point, and the second portion lies in $g\tau_{e_i}\mathcal{P}^\circ$
- This can be done G equivariantly.

Cayley Graphs - 2

Embedded Cayley graph examples:



Cayley Graphs - 3

Abstract Cayley graph (group theory):

- Vertices: The elements of G.
- Edges: for each e_i and $g \in G$, construct the edge $(e_i, g, g\tau_{e_i})$ = (edge crossed, start polygon, ending polygon)
- Slightly different than standard definition - we can have loops and multiple arcs between vertices.
- G acts on the left: for $h \in G$, vertices $g \to hg$, edges $(e_i, g, g\tau_{e_i}) \to (e_i, hg, hg\tau_{e_i})$.
- If desired, we may colour the edges with a different colour for each edge type e_i.

Sample Result

There are many more questions than results, but at least we have the following:

Theorem

Suppose that S_1 and S_2 are modular companions that both carry equivariant tilings, obtained by lifting the same cut system on their common quotient $S_1/G = S_2/G$. Then, the two surfaces are conformally equivalent (respecting the tilings) if and only if the abstract Cayley graphs are isomorphic, respecting edge types:

Questions - 1

Here are some questions, on which we are working, for which there are partial answers, or at least a plan.

- Determine the number of components lying over the space of quotients via $q : \mathcal{S}_{G,\mathfrak{s}} \to \mathcal{B}_{G,\mathfrak{s}}$. Partially done for 4 branch points.
- For each component noted above, determine the number of modular companions lying over a common quotient S/G. Partially done for 4 branch points.
- Describe the topology of the components noted above. Partially done for 4 branch points.

Questions - 2

- Find necessary and sufficient conditions on the set of crossover transformations $(\tau_{e_i} : i = 1, \ldots, 2k)$ to form a valid rule book (should be easy).
- A partial isometry is a map from a connected union of polygons in S_1 to another such union in a modular companion S_2. This can be measured using the abstract Cayley graph. What can be said?

Group action strata: Examples

In the table below we give information for the covering map:

$$q : \mathcal{S}_{G,\mathfrak{s}} \to \mathcal{B}_{G,\mathfrak{s}}$$

- 3rd column: signature, 4th column: genus of S, 5'th column total #surfaces lying over generic S/G.
- 6'th column is a list of #Modular Companions when restricted to a component of the cover q (action stratum).

G	$\|G\|$	$\mathfrak{s}$	σ	#surfs	#ModComp
$Sym(3)$	6	$(2,2,3,3)$	2	2	$(1,1)$
$Alt(5)$	60	$(2,2,3,3)$	11	9	(9)
$Alt(5)$	60	$(2,3,3,5)$	20	20	(20)
$Alt(5)$	60	$(5,5,5,5)$	37	47	$(6,10,15,16)$
$PSL(2,7)$	168	$(2,2,3,3)$	29	15	(15)
$PSL(2,7)$	168	$(7,7,7,7)$	121	95	$(6,7,16,24,42)$
$PSL(2,11)$	660	$(5,5,5,5)$	397	4906	not done

References

- G. Bartolini, A. Costa, M. Izquierdo, *On isolated strata of p-gonal Riemann surfaces in the branch locus of moduli spaces*, Albanian J. Math. **6**, no, 1, (2012) 11–19.
- G. Bartolini, A. Costa, M. Izquierdo, *On automorphism groups of cyclic p-gonal Riemann surfaces*, J. Symbolic Comput. **57**, (2013) 61–69.
- A. Costa, M. Izquierdo, H. Parlier, *Connecting p-gonal loci in the compactification of moduli space*, arXiv:1305.0284v2.
- S.A. Broughton, *The equisymmetric stratification of the moduli space and the Krull dimension of the mapping class group*, Topology and its Applications **37** (1990), 101–113.

Thank You!

Any questions?