Plato's Criticism of the Geometers in his Circle–Evidence About the History of Greek Mathematics from Plutarch

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Overview

1. Some personal remarks
2. Plutarch's accounts of Plato's criticism
3. Just what was Plato criticizing?
4. Historical context – what we know about this

- I've always been interested in the history of mathematics (in addition to my nominal specialties in algebraic geometry/ coding theory, etc.)
- Have been taking Greek and Latin courses with wonderfully welcoming colleagues in HC's Classics department
- Idea is to be able to engage with original texts on their own terms
- The subject for today comes from a paper that started out life as an assignment for Prof. Tom Martin's Plutarch seminar in Fall 2016 – thanks to him for much encouragement and guidance!

Plutarch

- Plutarch of Chaeronea (ca. 45 - ca. 120 CE)
- He records that he studied philosophy and mathematics in Athens at the Academy (w/successors of Plato)
- His prolific writings reveal a strong connection with Platonic traditions
- We would call him an essayist and biographer – his best known work today is certainly his *Parallel Lives* of illustrious Greeks and Romans
- But another extensive category of his writing has also survived – devoted more to philosophy and ethics, but with fascinating historical details at times

A first passage

- Comes from his *Life of Marcellus*
- Context: a discussion of the geometrical and mechanical work of Archimedes and the tradition that King Hiero of Syracuse persuaded him to take up mechanics to design engines of war in defence of his native city-state
- Marcellus was the commmander of the Roman forces in the siege of Syracuse in 212 BCE during which Archimedes was killed
- We’ll consider the well-known translation by Bernadotte Perrin (Loeb Classical Library edition of Plutarch)

Quotation, part I

"For the art of mechanics, now so celebrated and admired, was first originated by Eudoxus and Archytas, who embellished geometry with subtleties, and gave to problems incapable of proof by word and diagram, a support derived from mechanical illustrations that were patent to the senses. For instance in solving the problem of finding two mean proportional lines[1]*, a necessary requisite for many geometrical figures, both mathematicians had recourse to mechanical arrangements*[2]

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[1] I will explain this presently

[2] "κατασκευάς" – "constructions" would be another, perhaps better, translation here.

Quotation, part II

. . . adapting to their purposes certain intermediate portions of curved lines and sections.[3] *But Plato was incensed at this, and inveighed against them as corrupters and destroyers of the pure excellence of geometry, which thus turned her back upon the incorporeal things of abstract thought and descended to the things of sense, making use, moreover, of objects which required much mean and manual labor. For this reason, mechanics was made entirely distinct from geometry, and ... came to be regarded as one of the military arts."*

[3] "μεσογράφους τινὰς ἀπὸ καμπύλων καὶ τμημάτων μεθαρμόζοντες" – better: "adapting to their purposes mean proportionals found from curved lines and sections."

A second passage

- Comes from a section of his *Moralia* known as the *Quaestiones Convivales*, or “Table Talk”
- Presented as a record of conversation at a *sumposion*, or drinking party, arranged by Plutarch for a group of guests
- Philosophical questions are always debated
- The rationale for this: in “... our entertainments we should use learned and philosophical discourse ...” so that even if the guests become drunk, “... every thing that is brutish and outrageous in it [the drunkenness] is concealed ... "
- In other words, to keep your next party from degenerating into a drunken brawl, have your guests converse about Plato!

The role of the study of geometry

- One guest brings up the phrase "God always geometrizes" – he thinks it sounds like something Plato would have said.
- A second guest: Plato certainly said geometry is "... taking us away from the sensible and turning us back to the eternal nature we can perceive with our minds, whose contemplation is the goal of philosophy ... . Therefore even Plato himself strongly criticized Eudoxus, Archytas, and Menaechmus for attempting to reduce the duplication of the cube to tool-based and mechanical constructions, just as though they were trying, in an unreasoning way, to take two mean proportionals in continued proportion any way that they might ... ."[4]

[4] My translation – surprisingly technical(?) for a drinking party, don't you think!

- Plato's objection refers specifically to use of mechanical ideas, tools, or sensory data in pure geometry
- Book VII of the *Republic*: Plato has Socrates say in reference to geometry that "... it is the knowledge of that which always is, and not of a something which at some time comes into being and passes away. ... [I]t would tend to draw the soul to truth, and would be productive of a philosophical attitude of mind, directing upward the faculties that are now wrongly turned downward" (note echo in Plutarch's dinner conversation!)
- *Interesting sidelight:* In his *Memorabilia*, Xenophon has *his* Socrates say that practical geometry of measurement and apportionment is important and men should be able to demonstrate the correctness of their work, but he cannot see the usefulness of higher geometry(!)

When is a construction "mechanical" or "tool-based" or "reliant on the senses?"

- One way – use of *actual physical tools* (we'll see an example shortly)
- N.B. Euclidean straightedge and compass are "exempted" here of course – they are idealized constructs of the mind
- Use of motion or change over time – even the grammatical construction typically used in Greek to describe geometric constructions (e.g. γεγράφθω – "let it have been drawn") seems to emphasize that the figure or diagram *has been constructed* as a whole – a *static* conception
- Any use of sensory data to *approximate* a length or angle

The main actors

- Eudoxus of Cnidus (409–356 BCE), Archytas of Tarentum (428–347 BCE), and Menaechmus of Alopeconnesus (380–320 BCE)
- Three of the most accomplished Greek mathematicians active in the 4th century BCE.
- Archytas is often identified as a Pythagorean and there are traditions that Eudoxus was a pupil of his and Menaechmus was a pupil of Eudoxus.
- All three associated with Plato and his Academy in Athens
- Source: commentary on Book I of Euclid's *Elements* by Proclus (though the fact that Proclus is writing $\sim$ 800 years later raises the question of how reliable his information is).

Previous work on duplication of the cube

- Hippocrates of Chios (ca. 470–ca. 410 BCE)
- Given AB and GH, CD and EF are *two mean proportionals in continued proportion* if

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{GH}.$$

- Hippocrates' contribution: if $GH = 2AB$, then $CD^3 = 2AB^3$.
- In other words, if AB is the side of the original cube, then CD is the side of the cube with twice the volume.
- Geometric construction of the two mean proportionals was still an open question but this gave a way to attack the duplication of the cube; all later work started from this reduction.

Eutocius' catalog

- Plutarch does not say how Eudoxus, Archytas, or Menaechmus actually approached duplicating the cube.
- However, detailed accounts of the contributions of Archytas and Menaechmus and many others have survived – clearly a formative chapter in history of Greek mathematics
- Most importantly, a commentary on Archimedes' *On the Sphere and the Cylinder* by Eutocius of Ascalon (ca. 480 – ca. 540 CE. Note: $\sim$ 900 years after the fact!)
- Eutocius' includes detailed information about the approaches of Archytas and Menaechmus, but he does not present Eudoxus' solution "by means of curved lines" (he thinks the surviving accounts he has are corrupt).

Clearest case of what Plato seems to have had in mind

- Due to Eratosthenes of Cyrene (276 – 194 BCE)
- Eutocius includes a purported letter to King Ptolemy III of Egypt with a summary of earlier work and Eratosthenes' own solution making use of an instrument he dubbed the *mesolabe*, or "mean-taker"
- The purpose of the letter is essentially to claim the superiority of Eratosthenes' tool-based mechanical method for practical use. It was dismissed as a forgery by some 19th and early 20th century historians, but more recently, the tide of opinion has seemingly changed – consensus seems to be it should be accepted as authentic

A definitely "mechanical" solution based on sense data

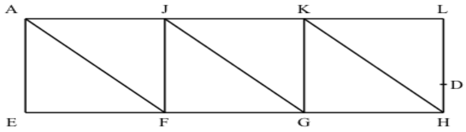

Figure: The *mesolabe* in original position.

Eratosthenes' solution, cont.

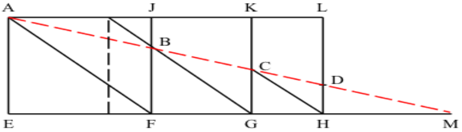

Figure: Using his or her senses and trial and error, the geometer maneuvers the left and right panels until this configuration with A, B, C, D collinear is reached. By similar triangles, $\frac{AE}{BF} = \frac{BF}{CG} = \frac{CG}{DH}$.

Archytas configurations

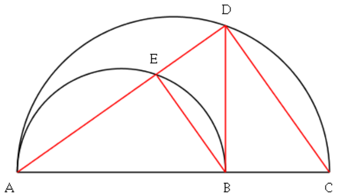

Figure: *AEB* and *ADC* are two semicircles tangent at *A*; *BD* is tangent to the smaller semicircle at *B*. Hence ΔBAE, ΔCAD, ΔDBE, ΔCDB and ΔDAB are all similar and $\frac{AE}{AB} = \frac{AB}{AD} = \frac{AD}{AC}$.

Finding such a configuration – a naive approach

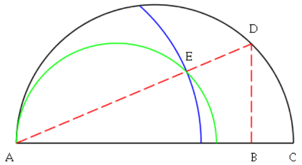

Figure: Given $AE < AC$, want E on the blue arc. Through each such point there is exactly one semicircle tangent at A, shown in green. AE meets outer semicircle at D and B is foot of the perpendicular from D. Increasing $\angle CAE$, BD will meet inner semicircle. Hence, by continuity, there exists E yielding an Archytas configuration.

Archytas' solution and modern interpretations

- What Eutocius said that Archytas actually did here has been interpreted in a number of ways
- T.L. Heath's influential history interprets Archytas' solution as a bold foray into solid geometry whereby a suitable E is found by intersecting three surfaces in three dimensions (a cylinder, a cone and a degenerate semi-torus–the surface of revolution generated by rotating the semicircle with diameter AC about its tangent line at A).
- Heath characterizes this solution as "the most remarkable of all" discussed by Eutocius because of the sophisticated use of three-dimensional geometry he sees in it.
- Similarly, Knorr calls it a "stunning *tour de force* of stereometric insight."

But is that an anachronistic reading?

- It's not easy to see all of the elements of Heath's reconstruction in the actual text
- While a (semi-)cylinder and a cone are explicitly mentioned, the semi-torus surface of revolution is not.
- Moreover, even there, the cone and its properties are not really used in the proof; it seems to be included more for the purposes of visualization and to show how an exact solution could be specified without recourse to approximation.
- Eutocius *does not* single out Archytas' solution as the "most remarkable" in any way

Another possible reconstruction

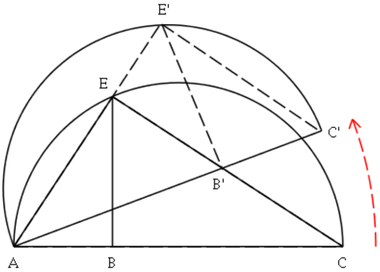

Figure: Another suggestion from a recent article by Masià. The rotation is continued until B' lies on CE.

A "mechanical" solution?

- The kinematic nature of the naive solution and also Masià's suggestion seem to match up pretty well with one possible interpretation of Plutarch's account of Plato's criticism here(!)
- One could also easily imagine a device to carry out the planar rotation described before
- My reading: Heath's version (intersection of three surfaces in three-dimensions) seems *both* closer to the static ideal of Plato's "take" on Greek geometry, *and (ironically)* technologically (far) too advanced for the time of Archytas, when geometry in three dimensions was in its very infancy – quite mysterious.

The work of Menaechmus

The approach attributed by Eutocius to Menaechmus is even more problematic although it was evidently extremely influential for later Greek geometry. Given line segments of lengths a, b, finding the two mean proportionals in continued proportion means finding x, y to satisfy:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

Hence, using coordinate geometry (very anachronistically), we see the solution will come from the point of intersection of the parabola $ay = x^2$ and the hyperbola $xy = ab$, or one of the points of intersection of the two parabolas $ay = x^2$ and $bx = y^2$.

Questions

- As beautiful as this is, has Eutocius preserved a historically accurate account of Menaechmus' work?
- In particular, could Menaechmus have recognized that he was dealing with a conic section from $ay = x^2$, the a, y, x would have represented line segments and each side would have represented an area?
- None of Menaechmus' own writings have survived. Suspiciously, the discussion of his work in Eutocius uses the terminology for conic sections introduced much after the time of Menaechmus himself by Apollonius of Perga (262–190 BCE).
- Apollonius' work does provide exactly the point of view needed to connect sections of a cone with equations such as $ay = x^2$ or $xy = ab$.

More questions and a few answers (?)

- Apollonius' terminology and conceptual framework for conics seems to have been developed by analogy with constructions in the *application of areas* (a technique that Menaechmus would have known well)
- A connection between Menaechmus and the later theory of conics undoubtedly exists. But did he have a theory of conics (i.e. curves described as sections of cones)?
- Seems much more likely (to me, and to many other recent historians) that the theory of conics grew out of what Menaechmus did, but that he probably did not have the whole picture himself(!)
- Whatever sources Eutocius had for this reworked Menaechmus in the light of later developments.

An interesting sidelight

How might the adjectives "mechanical" or "instrument-" or "tool-based" apply to what is attributed to Menaechmus by Eutocius? The conic sections apart from the circle cannot be constructed as whole curves using only the Euclidean tools and other sorts of devices would be needed to produce them. Eutocius' discussion does include a final comment that "the parabola is drawn by the compass invented by our teacher the mechanician Isidore of Miletus ... ." Isidore (442–537 CE) was an architect, one of the designers of the *Hagia Sophia* in Constantinople, and thus this note is surely an interpolation, not a part of the older source Eutocius was using to produce this section of his commentary.

Conclusions

- Plutarch was certainly in contact with the Platonic tradition, but from the work of Archytas and Menaechmus and later Archimedes, Apollonius and others, if something like Plato's criticism actually happened at this point in history, then its effect on Greek mathematics was minimal
- An openness to mechanical techniques can be seen in many authors, perhaps preeminently Archimedes
- Heron of Alexandria (ca. 10 – ca. 70 CE) gives another way to find the two mean proportionals in his Βελοποιϊκά, a treatise on the design of siege engines and artillery(!)
- While it drew on philosophy for its norms of logical rigor, I would agree with Knorr that mathematics had in essence emerged as an independent subject in its own right

Conclusions, cont.

Elsewhere in the *Republic*, Plato's Socrates pokes fun at geometers:

> *"Their language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge."*

Thinking about the implications of this and what Greek mathematicians were doing, it seems doubtful that Plato's ideas about the proper methods or goals of mathematics carried much real weight for many of its actual practitioners.

References

[1] T. L. Heath, *A History of Greek Mathematics*, vol. I, Dover, NY 1981; reprint of the original ed., Oxford University Press, 1921.

[2] Wilbur Knorr, *The Ancient Tradition of Geometric Problems*, Dover, NY, 1993; corrected reprint of original ed., Birkhäuser, Boston, 1986.

[3] Reviel Netz, *The Works of Archimedes, Volume 1: The Two Books On the Sphere and the Cylinder*, Cambridge University Press NY, 2004.

[4] Plutarch, *Quaestiones Convivales*, available at http://data.perseus.org/texts:um:cts:greekLit:tlg0007.tlg112.perseus-grc1