Toric Varieties

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Chapter 1

Affine Toric Varieties

0. Background: Affine Varieties

We begin with the algebraic geometry needed for our study of affine toric varieties. Our discussion assumes Chapters 1–5 and 9 of [**6**].

Coordinate Rings. An ideal $I \subseteq S = \mathbb{C}[x_0, \ldots, x_n]$ gives an affine variety

$$\mathbf{V}(I) = \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in I\}$$

and an affine variety $V \subseteq \mathbb{C}^n$ gives the ideal

$$\mathbf{I}(V) = \{f \in S \mid f(p) = 0 \text{ for all } p \in V\}.$$

By the Hilbert Basis Theorem, an affine variety V is defined by the vanishing of finitely many polynomials in S, and for any ideal I, the Nullstellensatz tells us that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = \{f \in S \mid f^k \in I \text{ for some } k \geq 1\}$ since $\mathbb{C}$ is algebraically closed. The most important algebraic object associated to V is its *coordinate ring*

$$\mathbb{C}[V] = S/\mathbf{I}(V).$$

Elements of $\mathbb{C}[V]$ can be interpreted as the $\mathbb{C}$-valued polynomial functions on V. Note that $\mathbb{C}[V]$ is a $\mathbb{C}$-algebra, meaning that its vector space structure is compatible with its ring structure. Here are some basic facts about coordinate rings:

- $\mathbb{C}[V]$ is an integral domain $\Leftrightarrow$ $\mathbf{I}(V)$ is a prime ideal $\Leftrightarrow$ V is irreducible.
- Polynomial maps (also called *morphisms*) $F : V_1 \to V_2$ between affine varieties correspond to $\mathbb{C}$-algebra homomorphisms $F^* : \mathbb{C}[V_2] \to \mathbb{C}[V_1]$, where $F^*(g) = g \circ F$ for $g \in \mathbb{C}[V_2]$.
- Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic $\mathbb{C}$-algebras.

- A point p of an affine variety V gives the maximal ideal
$$\{f \in \mathbb{C}[V] \mid f(p) = 0\} \subseteq \mathbb{C}[V],$$
and all maximal ideals of $\mathbb{C}[V]$ arise this way.

Coordinate rings of affine varieties can be characterized as follows (Exercise 0.1).

Lemma 0.1. *A $\mathbb{C}$-algebra R is isomorphic to the coordinate ring of an affine variety if and only if R is a finitely generated $\mathbb{C}$-algebra with no nonzero nilpotents, i.e., if $f \in R$ satisfies $f^m = 0$, then $f = 0$.* □

To emphasize the close relation between V and $\mathbb{C}[V]$, we sometimes write

$$V = \operatorname{Spec}(\mathbb{C}[V]). \tag{0.1}$$

This can be made canonical by identifying V with the set of maximal ideals of $\mathbb{C}[V]$ via the fourth bullet above. More generally, one can take any commutative ring R and define the *affine scheme* $\operatorname{Spec}(R)$. The general definition of Spec uses all prime ideals of R, not just the maximal ideals as we have done. Thus some authors would write (0.1) as $V = \operatorname{Specm}(\mathbb{C}[V])$, the maximal spectrum of $\mathbb{C}[V]$). Readers wishing to learn more about affine schemes should consult [**9**] and [**13**].

The Zariski Topology. An affine variety $V \subseteq \mathbb{C}^n$ has two topologies we will use. The first is the *classical topology*, induced from the usual topology on $\mathbb{C}^n$. The second is the *Zariski topology*, where the Zariski closed sets are subvarieties of V (meaning affine varieties of $\mathbb{C}^n$ contained in V) and the Zariski open sets are their complements. Since subvarieties are closed in the classical topology (polynomials are continuous), Zariski open subsets are open in the classical topology.

Given a subset $S \subseteq V$, its closure $\overline{S}$ in the Zariski topology is the smallest subvariety of V containing S. We call $\overline{S}$ the *Zariski closure* of S. It is easy to give examples where this differs from the closure in the classical topology.

Affine Open Subsets and Localization. Some Zariski open subsets of an affine variety V are themselves affine varieties. Given $f \in \mathbb{C}[V] \setminus \{0\}$, let

$$V_f = \{p \in V \mid f(p) \neq 0\} \subseteq V.$$

Then V_f is Zariski open in V and is also an affine variety, as we now explain.

Let $V \subseteq \mathbb{C}^n$ have $\mathbf{I}(V) = \langle f_1, \ldots, f_s\rangle$ and pick $g \in \mathbb{C}[x_1, \ldots, x_n]$ representing f. Then $V_f = V - \mathbf{V}(g)$ is Zariski open in V. Now consider a new variable y and let $W = \mathbf{V}(f_1, \ldots, f_s, 1 - gy) \subseteq \mathbb{C}^n \times \mathbb{C}$. Since the projection map $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps W bijectively onto V_f, we can identify V_f with the affine variety $W \subseteq \mathbb{C}^n \times \mathbb{C}$.

When V is irreducible, the coordinate ring of V_f is easy to describe. Let $\mathbb{C}(V)$ be the field of fractions of the integral domain $\mathbb{C}[V]$. Recall that elements of $\mathbb{C}(V)$

give rational functions on V. Then let

$$\mathbb{C}[V]_f = \{g/f^\ell \in \mathbb{C}(V) \mid g \in \mathbb{C}[V],\ \ell \geq 0\}. \tag{0.2}$$

In Exercise 0.2 you will prove that $\operatorname{Spec}(\mathbb{C}[V]_f)$ is the affine variety V_f.

Example 0.2. The n-dimensional torus is the affine open subset

$$(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathbf{V}(x_1 \cdots x_n) \subseteq \mathbb{C}^n,$$

with coordinate ring

$$\mathbb{C}[x_0, \ldots, x_n]_{x_1 \cdots x_n} = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Elements of this ring are called *Laurent polynomials*. ◇

The ring $\mathbb{C}[V]_f$ from (0.2) is an example of *localization*. In Exercise 0.2 you will show how to construct this ring for all affine varieties, not just irreducible ones. The general concept of localization is discussed in standard texts in commutative algebra such as [**2**, Chapter 3] and [**8**, Chapter 2].

Normal Affine Varieties. Let R be an integral domain with field of fractions K. Then R is *normal* if every element of K which is integral over R (meaning that it is a root of a monic polynomial in $R[x]$) actually lies in R. For example, any UFD is normal (Exercise 0.3).

Definition 0.3. An irreducible affine variety V is ***normal*** if its coordinate ring $\mathbb{C}[V]$ is normal.

For example, $\mathbb{C}^n$ is normal since its coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ is a UFD and hence normal. Here is an example of a non-normal affine variety.

Example 0.4. Let $C = \mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$. This is an irreducible plane curve with a cusp at the origin. It is easy to see that $\mathbb{C}[C] = \mathbb{C}[x,y]/\langle x^3 - y^2\rangle$. Now let $\bar{x}$ and $\bar{y}$ be the cosets of x and y in $\mathbb{C}[C]$ respectively. This gives $\bar{y}/\bar{x} \in \mathbb{C}(C)$. A computation shows that $\bar{y}/\bar{x} \notin \mathbb{C}[C]$ and that $(\bar{y}/\bar{x})^2 = \bar{x}$. Consequently $\mathbb{C}[C]$ and hence C are not normal.

We will see below that C is an affine toric variety. ◇

An irreducible affine variety V has a *normalization* defined as follows. Let

$$\mathbb{C}[V]' = \{\alpha \in \mathbb{C}(V) : \alpha \text{ is integral over } \mathbb{C}[V]\}.$$

We call $\mathbb{C}[V]'$ the *integral closure* of $\mathbb{C}[V]$. One can show that $\mathbb{C}[V]'$ is normal and (with more work) finitely generated as a $\mathbb{C}$-algebra (see [**8**, Corollary 13.13]). This gives the normal affine variety

$$V' = \operatorname{Spec}(\mathbb{C}[V]')$$

We call V' the *normalization* of V. The natural inclusion $\mathbb{C}[V] \subseteq \mathbb{C}[V]' = \mathbb{C}[V']$ corresponds to a map $V' \to V$. This is the *normalization map*.

Example 0.5. We saw in Example 0.4 that the curve $C \subseteq \mathbb{C}^2$ defined by $x^3 = y^2$ has elements $X, Y \in \mathbb{C}[C]$ such that $Y/X \notin \mathbb{C}[C]$ is integral over $\mathbb{C}[C]$. In Exercise 0.4 you will show that $\mathbb{C}[\bar{y}/\bar{x}] \subseteq \mathbb{C}(C)$ is the integral closure of $\mathbb{C}[C]$ and that the normalization map is the map $\mathbb{C} \to C$ defined by $t \mapsto (t^2, t^3)$. ◇

At first glance, the definition of normal does not seem very intuitive. Once we enter the world of toric varieties, however, we will see that normality has a very nice combinatorial interpretation and that the nicest toric varieties are the normal ones. We will also see that normality leads to a nice theory of divisors.

In Exercise 0.5 you will prove some properties of normal domains that will be used in §3 when we study normal affine toric varieties.

Smooth Points of Affine Varieties. In order to define a smooth point of an affine variety V, we first need to define *local rings* and *Zariski tangent spaces*. For simplicity, we assume that V is irreducible. The *local ring* of V at p is

$$\mathscr{O}_{V,p} = \{f/g \in \mathbb{C}(V) \mid f, g \in \mathbb{C}[V] \text{ and } g(p) \neq 0\}.$$

Thus $\mathscr{O}_{V,p}$ consists of all rational functions on V that are defined at p. Inside of $\mathscr{O}_{V,p}$ we have the maximal ideal

$$\mathfrak{m}_{V,p} = \{\varphi \in \mathscr{O}_{V,p} \mid \varphi(p) = 0\}.$$

One can prove without difficulty that $\mathfrak{m}_{V,p}$ is the unique maximal ideal of $\mathscr{O}_{V,p}$. Rings with this property are called *local rings*.

The *Zariski tangent space* of V at p is defined to be

$$T_p(V) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2, \mathbb{C}).$$

In Exercise 0.6 you will verify that $\dim T_p(\mathbb{C}^n) = n$ for every $p \in \mathbb{C}^n$. According to [**13**, p. 32], we can compute the Zariski tangent space of a point in an affine variety as follows.

Lemma 0.6. *Let $V \subseteq \mathbb{C}^n$ be a affine variety and let $p \in V$. Also assume that $\mathbf{I}(V) = \langle f_1, \ldots, f_s\rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$. For each i, let*

$$d_p(f_i) = \frac{\partial f_i}{\partial x_1}(p)\, x_1 + \cdots + \frac{\partial f_i}{\partial x_n}(p)\, x_n.$$

The the Zariski tangent $T_p(V)$ is isomorphic to the subspace of $\mathbb{C}^n$ defined by the equations $d_p(f_1) = \cdots = d_p(f_s) = 0$. In particular, $\dim T_p(V) \leq n$. □

Definition 0.7. A point p of an affine variety V is ***smooth*** or ***nonsingular*** if $\dim T_p(V) = \dim V$. The point p is ***singular*** if it is not smooth. Finally, V is ***smooth*** if every point of V is smooth.

Since $\dim T_p(\mathbb{C}^n) = n$ for every $p \in \mathbb{C}^n$, we see that $\mathbb{C}^n$ is smooth. For an irreducible affine variety $V \subseteq \mathbb{C}^n$ of dimension d, fix $p \in V$ and write $\mathbf{I}(V) =$

$\langle f_1, \dots, f_s \rangle$. Using Lemma 0.6, it is straightforward to show that V is smooth at p if and only if the Jacobian matrix

$$J_p(f_1, \dots, f_s) = \Big(\frac{\partial f_i}{\partial x_j}(p)\Big)_{1 \le i \le s, 1 \le j \le n} \tag{0.3}$$

has rank $n - d$ (Exercise 0.7). Here is a simple example.

Example 0.8. As noted in Example 0.4, the plane curve C defined by $x^3 = y^2$ has $\mathbf{I}(C) = \langle x^3 - y^2 \rangle \subseteq \mathbb{C}[x, y]$. A point $p = (a, b) \in C$ has Jacobian

$$J_p = (3a^2, 2b),$$

so the origin is the only singular point of C.

Since $T_p(V) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2, \mathbb{C})$, we see that V is smooth at p when $\dim V$ equals the dimension of $\mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2$ as a vector space over $\mathscr{O}_{V,p}/\mathfrak{m}_{V,p}$. In terms of commutative algebra, this means that $p \in V$ is smooth if and only if $\mathscr{O}_{V,p}$ is a *regular local ring*. See [**2**, p. 123] or [**8**, 10.3].

We can relate smoothness and normality as follows.

Proposition 0.9. *A smooth irreducible affine variety V is normal.*

Proof. As noted above, the smoothness of V implies that $\mathscr{O}_{V,p}$ is a regular local ring for all $p \in V$. A powerful result of commutative algebra states that a regular local ring is a UFD (see [**8**, Theorem 19.19]). Hence $\mathscr{O}_{V,p}$ is normal for all p. In §0 of Chapter 3 we will see that

$$\mathbb{C}[V] = \bigcap_{p \in V} \mathscr{O}_{V,p},$$

which by Exercise 0.5 implies that $\mathbb{C}[V]$ is normal. □

The converse of Propostion 0.9 can fail. We will see in §3 that the affine variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is normal, yet $\mathbf{V}(xy - zw)$ is singular at the origin.

Products of Affine Varieties. Given affine varieties V_1 and V_2, there are several ways to show that the cartesian product $V_1 \times V_2$ is an affine variety. The most direct way is to proceed as follows. Let $V_1 \subseteq \mathbb{C}^m = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_m])$ and $V_2 \subseteq \mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[y_1, \dots, y_n])$. Take $\mathbf{I}(V_1) = \langle f_1, \dots, f_s \rangle$ and $\mathbf{I}(V_2) = \langle g_1, \dots, g_t \rangle$. Since the f_i and g_j depend on separate sets of variables, it follows that

$$V_1 \times V_2 = \mathbf{V}(f_1, \dots, f_s, g_1, \dots, g_t) \subseteq \mathbb{C}^{m+n}$$

is an affine variety.

A fancier method is to use the mapping properties of the product. This will also give an instrinic description of its coordinate ring. Given V_1 and V_2 as above,

$V_1 \times V_2$ should be an affine variety with projections $\pi_i : V_1 \times V_2 \to V_i$ such that whenever we have a diagram

$$\begin{tikzcd}
W \arrow[drr, bend left, "\phi_1"] \arrow[ddr, bend right, "\phi_2"'] \arrow[dr, dotted, "\nu"] & & \\
& V_1 \times V_2 \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & V_1 \\
& V_2 &
\end{tikzcd}$$

where $\phi_i : W \to V_i$ are morphisms from an affine variety W, there should be a unique morphism ν (the dotted arrow) that makes the diagram commute, i.e., $\pi_i \circ \nu = \phi_i$. For the coordinate rings, this means that whenever we have a diagram

$$\begin{tikzcd}
& \mathbb{C}[V_2] \arrow[d, "\pi_2^*"'] \arrow[ddr, bend left, "\phi_2^*"] & \\
\mathbb{C}[V_1] \arrow[r, "\pi_1^*"] \arrow[drr, bend right, "\phi_1^*"'] & \mathbb{C}[V_1 \times V_2] \arrow[dr, dotted, "\nu^*"] & \\
& & \mathbb{C}[W]
\end{tikzcd}$$

with $\mathbb{C}$-algebra homomorphisms $\phi_i^* : \mathbb{C}[V_i] \to \mathbb{C}[W]$, there should be a unique $\mathbb{C}$-algebra homomorphism ν^* (the dotted arrow) that makes the diagram commute. By the universal mapping property of the *tensor product* of $\mathbb{C}$-algebras, there is exactly one algebra that does this (up to isomorphism), namely $\mathbb{C}[V_1] \otimes_{\mathbb{C}} \mathbb{C}[V_2]$. This is a finitely generated $\mathbb{C}$-algebra with no nilpotents, so it is the coordinate ring of an affine variety by Lemma 0.1. We could take this as the definition of $V_1 \times V_2$.

Example 0.10. Let V be an affine variety. Since $\mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_n])$, the product $V \times \mathbb{C}^n$ has coordinate ring

$$\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_n] = \mathbb{C}[V][y_1, \ldots, y_n].$$

If V is contained in $\mathbb{C}^m$ with $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{C}[x_1, \ldots, x_m]$, it follows that

$$\mathbf{I}(V \times \mathbb{C}^n) = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n].$$

For later purposes, we also note that the coordinate ring of $V \times (\mathbb{C}^*)^n$ is

$$\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] = \mathbb{C}[V][y_1^{\pm 1}, \ldots, y_n^{\pm 1}]. \qquad \diamond$$

Given affine varieties V_1 and V_2, we note that the Zariski topology on $V_1 \times V_2$ is usually *not* the product of the Zariski topologies on V_1 and V_2.

Example 0.11. Consider $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. By definition, a basis for the product of the Zariski topologies consists of sets $U_1 \times U_2$ where U_i are Zariski open in $\mathbb{C}$. Such a set is the complement of a union of collections of "horizontal" and "vertical" lines in $\mathbb{C}^2$. This makes it easy to see that Zariski closed sets in $\mathbb{C}^2$ such as $\mathbf{V}(y - x^2)$ cannot be closed in the product topology. $\Diamond$

Exercises for §0.

0.1. Prove Lemma 0.1. Hint: You will need the Nullstellensatz.

0.2. Let R be a finitely generated $\mathbb{C}$-algebra without nilpotents as in Lemma 0.1 and let $f \in R$ be nonzero. The *localization* R_f consists of all formal expressions g/f^k, $g \in R$, modulo the equivalence relation

$$g/f^k \sim h/f^\ell \iff f^s(f^\ell g - f^k h) = 0 \text{ for some } s \geq 0.$$

(a) Show that R_f satisfies $\mathrm{Spec}(R_f) = \mathrm{Spec}(R)_f$.

(b) Show that R_f is given by (0.2) when R is an integral domain.

0.3. Prove that a UFD is normal.

0.4. In the setting of Example 0.5, show that $\mathbb{C}[\bar{y}/\bar{x}] \subseteq \mathbb{C}(C)$ is the integral closure of $\mathbb{C}[C]$ and that the normalization $\mathbb{C} \to C$ is defined by $t \mapsto (t^2, t^3)$.

0.5. In this exercise, you will prove some properties of normal domains need for §3.

(a) Let R be a normal domain with field of fractions K and fix $f \in R$. Prove that the localization R_f is normal.

(b) Let R_α. $\alpha \in A$, be normal domains with the same field of fractions K. Prove that the intersection $\bigcap_{\alpha\in A} R_\alpha$ is normal.

0.6. Prove that $\dim T_p(\mathbb{C}^n) = n$ for all $p \in \mathbb{C}^n$.

0.7. Use Lemma 0.6 to prove the claim made in the text that smoothness is determined by the rank of the Jacobian matrix (0.3).

1. Introduction to Affine Toric Varieties

We first discuss what we mean by "torus" and then explore various constructions of affine toric varieties.

The Torus. The affine variety $(\mathbb{C}^*)^n$ is a group under component-wise multiplication. A *torus* T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism. Associated to T are its *characters* and *one-parameter subgroups*. We discuss each of these briefly.

A *character* of a torus T is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. For example, $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ gives a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \cdots t_n^{a_n}. \tag{1.1}$$

One can show that *all* characters of $(\mathbb{C}^*)^n$ arise this way (see [**15**, §16]). Thus the characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$.

For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T. It is customary to say that $m \in M$ gives the character $\chi^m : T \to \mathbb{C}^*$.

We will need the following results concerning tori (see [**15**, §16] for proofs).

Proposition 1.1.

(a) *Let T_1 and T_2 be tori and let $\Phi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then the image of Φ is a torus and is closed in T_2.*

(b) *Let T be a torus and let $H \subseteq T$ be an irreducible subvariety of T that is a subgroup. Then H is a torus.* □

Now assume that a torus T acts linearly on a finite dimensional vector space W over $\mathbb{C}$, where the action of $t \in T$ on $w \in W$ is denoted $t \cdot w$. Given $m \in M$, we get the *eigenspace*

$$W_m = \{w \in W \mid t \cdot w = \chi^m(t)w \text{ for all } t \in T\}.$$

If $W_m \neq \{0\}$, then every $w \in W_m \setminus \{0\}$ is a simultaneous eigenvector for all $t \in T$, with eigenvalue given by $\chi^m(t)$.

Proposition 1.2. *In the above situation, we have $W = \bigoplus_{m \in M} W_m$.* □

This proposition is a sophisticated way of saying that a family of commuting diagonalizable linear maps can be simultaneously diagonalized.

A *one-parameter subgroup* of a torus T is a morphism $\chi : \mathbb{C}^* \to T$ that is a group homomorphism. For example, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by

$$\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n}). \tag{1.2}$$

All one-parameter subgroups of $(\mathbb{C}^*)^n$ arise this way (see [**15**, §16]). It follows that the group of one-parameter subgroups of $(\mathbb{C}^*)^n$ is naturally isomorphic to $\mathbb{Z}^n$. For an arbitrary torus T, the one-parameter subgroups form a free abelian group N of rank equal to the dimension of T. As with the character group, an element $u \in N$ gives the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T$.

There is a natural bilinear pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ defined as follows.

- (Intrinsic) Given a character χ^m and a one-parameter subgroup λ^u, the composition $\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ is character of $\mathbb{C}^*$, which is given by $t \mapsto t^\ell$ for some $\ell \in \mathbb{Z}$. Then $\langle m, u\rangle = \ell$.
- (Concrete) If $T = (\mathbb{C}^*)^n$ with $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, then one computes that

$$\langle m, u\rangle = \sum_{i=1}^n a_i b_i, \tag{1.3}$$

i.e., the pairing is the usual dot product.

It follows that the characters and one-parameter subgroups of a torus T form free abelian groups M and N of finite rank with a pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ that identifies N with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and M with $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. In terms of tensor products, one obtains a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T$ via $u \otimes t \mapsto \lambda^u(t)$. Hence it is customary to write a torus as T_N.

From this point of view, picking an isomorphism $T_N \simeq (\mathbb{C}^*)^n$ induces dual bases of M and N, i.e., isomorphisms $M \simeq \mathbb{Z}^n$ and $N \simeq \mathbb{Z}^n$ that turn characters into Laurent monomials (1.1), one-parameter subgroups into monomial curves (1.2), and the pairing into dot product (1.3).

The Definition of Affine Toric Variety. We now define the main object of study of this chapter.

Definition 1.3. An ***affine toric variety*** is an irreducible affine variety V containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an action of T_N on V.

Obvious examples of affine toric varieties are $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$. Here are some less trivial examples.

Example 1.4. The plane curve $C = \mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$ has a cusp at the origin. This is an affine toric variety with torus

$$C - \{0\} = C \cap (\mathbb{C}^*)^2 = \{(t^2, t^3) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^*,$$

where the isomorphism is $(t^2, t^3) \leftarrow\!\shortmid t$. Example 0.4 shows that C is a non-normal toric variety. $\Diamond$

Example 1.5. The variety $V = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is a toric variety with torus

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \mid t_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^3,$$

where the isomorphism is $(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \leftarrow\!\shortmid (t_1, t_2, t_3)$. We will see later that V is normal. $\Diamond$

Example 1.6. Consider the surface in $\mathbb{C}^{d+1}$ parametrized by the map

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^{d+1}$$

defined by $(s, t) \mapsto (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)$. Thus Φ is defined using all degree d monomials in s, t.

Let the coordinates of $\mathbb{C}^{d+1}$ be $x_0, \ldots, x_d$ and let $I \subseteq \mathbb{C}[x_0, \ldots, x_d]$ be the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix},$$

so $I = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1 \rangle$. In Exercise 1.1 you will verify that $\Phi(\mathbb{C}^2) = \mathbf{V}(I)$, so that $\widehat{C}_d = \Phi(\mathbb{C}^2)$ is an affine variety, and that $\mathbf{I}(\widehat{C}_d) = I$, so that I is the ideal of all polynomials vanishing on $\widehat{C}_d$. It follows that I is prime since V is irreducible by Proposition 1.8 below. The affine surface $\widehat{C}_d$ is called the *rational normal cone of degree* d and is an example of a *determinantal variety*. We will see below that I is an example of a toric ideal.

It is straightforward to show that $\widehat{C}_d$ is a toric variety with torus

$$\Phi((\mathbb{C}^*)^2) = \widehat{C}_d \cap (\mathbb{C}^*)^{d+1} \simeq (\mathbb{C}^*)^2.$$

We will study this example from the projective point of view in Chapter 2. ◇

We next explore three equivalent ways of constructing affine toric varieties.

Lattice Points. In this book, a *lattice* is a free abelian group of finite rank. Thus a lattice of rank n is isomorphic to $\mathbb{Z}^n$. For example, a torus T_N has lattices M (of characters) and N (of one-parameter subgroups).

Given a torus T_N with character lattice M, a set $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$ gives characters $\chi^{m_i} : T_N \to \mathbb{C}^*$. Then consider the map

$$\Phi_{\mathscr{A}} : T_N \longrightarrow \mathbb{C}^s \tag{1.4}$$

defined by

$$\Phi_{\mathscr{A}}(t) = \big(\chi^{m_1}(t), \ldots, \chi^{m_s}(t)\big) \in \mathbb{C}^s.$$

Definition 1.7. Given a finite set $\mathscr{A} \subseteq M$, the affine toric variety $Y_{\mathscr{A}}$ is defined to be the Zariski closure of the image of the map $\Phi_{\mathscr{A}}$ from (1.4).

This definition is justified by the following proposition.

Proposition 1.8. *Given $\mathscr{A} \subseteq M$ as above, let $\mathbb{Z}\mathscr{A} \subseteq M$ be the sublattice generated by $\mathscr{A}$. Then $Y_{\mathscr{A}}$ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathscr{A}$. In particular, the dimension of $Y_{\mathscr{A}}$ is the rank of $\mathbb{Z}\mathscr{A}$.*

Proof. The map (1.4) can be regarded as a map

$$\Phi_{\mathscr{A}} : T_N \longrightarrow (\mathbb{C}^*)^s$$

of tori. By Proposition 1.1, the image $T = \Phi_{\mathscr{A}}(T_N)$ is a torus that is closed in $(\mathbb{C}^*)^s$. The latter implies that $Y_{\mathscr{A}} \cap (\mathbb{C}^*)^s = T$ since $Y_{\mathscr{A}}$ is the Zariski closure of the image. It follows that the image is Zariski open in $Y_{\mathscr{A}}$. Furthermore, T is irreducible (it is a torus), so the same is true for its Zariski closure $Y_{\mathscr{A}}$.

We next consider the action of T. Since $T \subseteq (\mathbb{C}^*)^s$, an element $t \in T$ acts on $\mathbb{C}^s$ and takes varieties to varieties. Then

$$T = t \cdot T \subseteq t \cdot Y_{\mathscr{A}}$$

shows that $t \cdot Y_{\mathscr{A}}$ is a variety containing T. Hence $Y_{\mathscr{A}} \subseteq t \cdot Y_{\mathscr{A}}$ by the definition of Zariski closure. Replacing t with t^{-1} leads to $Y_{\mathscr{A}} = t \cdot Y_{\mathscr{A}}$, so that the action of T induces an action on $Y_{\mathscr{A}}$. We conclude that $Y_{\mathscr{A}}$ is an affine toric variety.

It remains to compute the character lattice $M(T)$ of T. Since $T = \Phi_{\mathscr{A}}(T_N)$, the map $\Phi_{\mathscr{A}}$ gives the commutative diagram

$$\begin{tikzcd} T_N \arrow[r, "\Phi_{\mathscr{A}}"] \arrow[dr, two heads] & (\mathbb{C}^*)^s \\ & T \arrow[u, hook] \end{tikzcd}$$

where $\twoheadrightarrow$ denotes a surjective map and $\hookrightarrow$ an injective map. This diagram of tori induces a commutative diagram of character lattices

$$\begin{tikzcd} M & \mathbb{Z}^s \arrow[l, "\widehat{\Phi}_{\mathscr{A}}"'] \arrow[d, two heads] \\ & M(T). \arrow[ul, hook] \end{tikzcd}$$

Since $\widehat{\Phi}_{\mathscr{A}} : \mathbb{Z}^s \to M$ takes the standard basis $e_1, \ldots, e_s$ to $m_1, \ldots, m_s$, the image of $\widehat{\Phi}_{\mathscr{A}}$ is $\mathbb{Z}\mathscr{A}$. By the diagram, we obtain $M(T) \simeq \mathbb{Z}\mathscr{A}$. Then we are done since the dimension of a torus equals the rank of its character lattice. □

In concrete terms, fix a basis of M, so that we may assume $M = \mathbb{Z}^n$. Then the s vectors in $\mathscr{A} \subseteq \mathbb{Z}^n$ can be regarded as the columns of an $n \times s$ matrix A with integer entries. In this case, the the dimension of $Y_{\mathscr{A}}$ is simply the rank of the matrix A.

We will see below that every affine toric variety is isomorphic to $Y_{\mathscr{A}}$ for some finite subset $\mathscr{A}$ of a lattice.

Toric Ideals. Let $Y_{\mathscr{A}} \subseteq \mathbb{C}^s = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_s])$ be the affine toric variety coming from a finite set $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$. We can describe the ideal $\mathbf{I}(Y_{\mathscr{A}}) \subseteq \mathbb{C}[x_1, \ldots, x_s]$ as follows. As in the proof of Proposition 1.8, (1.4) induces a map of character lattices

$$\widehat{\Phi}_{\mathscr{A}} : \mathbb{Z}^s \longrightarrow M$$

that sends the standard basis $e_1, \ldots, e_s$ to $m_1, \ldots, m_s$. Let L be the kernel of this map, so that we have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow M.$$

In down to earth terms, elements $\ell = (\ell_1, \ldots, \ell_s)$ of L satisfy $\sum_{i=1}^s \ell_i m_i = 0$ and hence record the linear relations among the m_i.

Given $\ell = (\ell_1, \ldots, \ell_s) \in L$, set

$$\ell_+ = \sum_{\ell_i > 0} \ell_i e_i \quad \text{and} \quad \ell_- = -\sum_{\ell_i < 0} \ell_i e_i.$$

Note that $\ell = \ell_+ - \ell_-$ and that $\ell_+, \ell_- \in \mathbb{N}^s$. It follows easily that the binomial

$$x^{\ell_+} - x^{\ell_-} = \prod_{\ell_i > 0} x_i^{\ell_i} - \prod_{\ell_i < 0} x_i^{-\ell_i}$$

vanishes on the image of (1.4) and hence on $Y_{\mathscr{A}}$ since $Y_{\mathscr{A}}$ is the Zariski closure of the image.

Proposition 1.9. *The ideal of the affine toric variety $Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ is*

$$\mathbf{I}(Y_{\mathscr{A}}) = \big\langle x^{\ell_+} - x^{\ell_-} \mid \ell \in L \big\rangle = \big\langle x^{\alpha} - x^{\beta} \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \big\rangle.$$

Proof. We leave it to the reader to prove equality of the two ideals on the right (Exercise 1.2). Let I_L denote this ideal and note that $I_L \subseteq \mathbf{I}(Y_{\mathscr{A}})$. We prove the opposite inclusion following [**26**, Lemma 4.1]. Pick a monomial order $>$ on $\mathbb{C}[x_1, \ldots, x_s]$ and an isomorphism $T_N \simeq (\mathbb{C}^*)^n$. Thus we may assume $M = \mathbb{Z}^n$ and the map $\Phi : (\mathbb{C}^*)^n \to \mathbb{C}^s$ is given by Laurent monomials t^{m_i} in variables $t_1, \ldots, t_n$. If $I_L \neq \mathbf{I}(Y_{\mathscr{A}})$, then we can pick $f \in \mathbf{I}(Y_{\mathscr{A}}) \setminus I_L$ with minimal leading monomial $x^{\alpha} = \prod_{i=1}^s x_i^{a_i}$. Rescaling if necessary, x^{α} becomes the leading term of f.

Since $f(t^{m_1}, \ldots, t^{m_s})$ is identically zero as a polynomial in $t_1, \ldots, t_n$, there must be cancellation involving the term coming from x^{α}. In other words, f must contain a monomial $x^{\beta} = \prod_{i=1}^s x_i^{b_i} < x^{\alpha}$ such that

$$\prod_{i=1}^{s} (t^{m_i})^{a_i} = \prod_{i=1}^{s} (t^{m_i})^{b_i}.$$

This implies that

$$\sum_{i=1}^{s} a_i m_i = \sum_{i=1}^{s} b_i m_i,$$

so that $\alpha - \beta = \sum_{i=1}^s (a_i - b_i) e_i \in L$. Then $x^{\alpha} - x^{\beta} \in I_L$ by the second description of I_L. It follows that $f - x^{\alpha} + x^{\beta}$ also lies in $\mathbf{I}(Y_{\mathscr{A}}) \setminus I_L$ and has strictly smaller leading term. This contradiction completes the proof. □

Given a finite set $\mathscr{A} \subseteq M$, there are several methods to compute the ideal $\mathbf{I}(Y_{\mathscr{A}}) = I_L$ of Proposition 1.9. For simple examples, the rational implicitization algorithm of [**6**, §3 of Chapter 3] can be used. It is also possible to compute I_L using a basis of L and ideal quotients (Exercise 1.3). Further comments on computing I_L can be found in [**26**, Chapter 12].

Inspired by Proposition 1.9, we make the following definition.

Definition 1.10. Let $L \subseteq \mathbb{Z}^s$ be a sublattice.

(a) The ideal $I_L = \big\langle x^{\alpha} - x^{\beta} \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \big\rangle$ is called a ***lattice ideal***.

(b) A prime lattice ideal is called a ***toric ideal***.

Since toric varieties are irreducible, the ideals appearing in Proposition 1.9 are toric ideals. Examples of toric ideals include:

Example 1.4 : $\langle x^3 - y^2 \rangle \subseteq \mathbb{C}[x, y]$
Example 1.5 : $\langle xz - yw \rangle \subseteq \mathbb{C}[x, y, z, w]$
Example 1.6 : $\langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \le i < j \le d-1 \rangle \subseteq \mathbb{C}[x_0, \ldots, x_d]$.

(The latter is the ideal of the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$.) In each example, we have a prime ideal generated by binomials. As we now show, such ideal are automatically toric.

Proposition 1.11. *An ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_s]$ is toric if and only if it is prime and generated by binomials.*

Proof. One direction is obvious. So suppose that I is prime and generated by binomials $x^{\alpha_i} - x^{\beta_i}$. Then observe that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is nonempty (it contains $(1, \ldots, 1)$) and is a subgroup of $(\mathbb{C}^*)^s$ (easy to check). Since $\mathbf{V}(I) \subseteq \mathbb{C}^s$ is irreducible, it follows that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is an irreducible subvariety of $(\mathbb{C}^*)^s$ that is also a subgroup. By Proposition 1.1, we see that $T = \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is a torus.

Projecting on the ith coordinate of $(\mathbb{C}^*)^s$ gives a character $T \hookrightarrow (\mathbb{C}^*)^s \to \mathbb{C}^*$, which by our usual convention we write as $\chi^{m_i} : T \to \mathbb{C}^*$ for $m_i \in M(T)$. It follows easily that $\mathbf{V}(I) = Y_{\mathscr{A}}$ for $\mathscr{A} = \{m_1, \ldots, m_s\}$, and since I is prime, we have $I = \mathbf{I}(Y_{\mathscr{A}})$ by the Nullstellensatz. Then I is toric by Proposition 1.9. $\square$

We will later see that all affine torics arise from toric ideals. For more on toric ideals and lattice ideals, the reader should consult [**17**, Chapter 7].

Affine Semigroups. A *semigroup* is a set S with an associative binary operation and an identity element. To be an *affine semigroup*, we further require that

- The binary operation on S is commutative. We will write the operation as $+$ and the identity element as 0. Thus a finite set $\mathscr{A} \subseteq \mathsf{S}$ gives
$$\mathbb{N}\mathscr{A} = \big\{ \textstyle\sum_{m \in \mathscr{A}} a_m m \mid a_m \in \mathbb{N} \big\} \subseteq \mathsf{S}.$$
- The semigroup is finitely generated, meaning that there is a finite set $\mathscr{A} \subseteq \mathsf{S}$ such that $\mathbb{N}\mathscr{A} = \mathsf{S}$.
- The semigroup can be embedded in a lattice M.

The simplest example of an affine semigroup is $\mathbb{N}^n \subseteq \mathbb{Z}^n$. More generally, given a lattice M and a finite set $\mathscr{A} \subseteq M$, we get the affine semigroup $\mathbb{N}\mathscr{A} \subseteq M$. Up to isomorphism, all affine semigroups are of this form.

Given an affine semigroup $\mathsf{S} \subseteq M$, the *semigroup algebra* $\mathbb{C}[\mathsf{S}]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication induced by the semigroup structure of S. To make this precise, we think of M as the character lattice of a torus T_N, so that $m \in M$ gives the character χ^m. Then

$$\mathbb{C}[\mathsf{S}] = \Big\{ \sum_{m \in \mathsf{S}} c_m \chi^m \mid c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m \Big\},$$

with multiplication induced by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

If $\mathsf{S} = \mathbb{N}\mathscr{A}$ for $\mathscr{A} = \{m_1, \ldots, m_s\}$, then $\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$.

Here are two basic examples.

Example 1.12. The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

$$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n],$$

where $x_i = \chi^{e_i}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. $\Diamond$

Example 1.13. If $e_1, \ldots, e_n$ is a basis of a lattice M, then M is generated by $\mathscr{A} = \{\pm e_1, \ldots, \pm e_n\}$ as an affine semigroup. Setting $t_i = \chi^{e_i}$ gives the Laurent polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

Using Example 0.2, one sees that $\mathbb{C}[M]$ is the coordinate ring of the torus T_N. $\Diamond$

Affine semigroup rings give rise to affine toric varieties as follows.

Proposition 1.14. *Let* $\mathsf{S} \subseteq M$ *be an affine semigroup.*

(a) $\mathbb{C}[\mathsf{S}]$ *is an integral domain and finitely generated as a* $\mathbb{C}$*-algebra.*

(b) $\operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ *is an affine toric variety whose torus has character lattice* $\mathbb{Z}\mathsf{S}$*, and if* $\mathsf{S} = \mathbb{N}\mathscr{A}$ *for a finite set* $\mathscr{A} \subseteq M$*, then* $\operatorname{Spec}(\mathbb{C}[\mathsf{S}]) = Y_{\mathscr{A}}$.

Proof. As noted above, $\mathscr{A} = \{m_1, \ldots, m_s\}$ implies $\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$, so $\mathbb{C}[\mathsf{S}]$ is finitely generated. Since $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[M]$ follows from $\mathsf{S} \subseteq M$, we see that $\mathbb{C}[\mathsf{S}]$ is an integral domain by Example 1.13.

Using $\mathscr{A} = \{m_1, \ldots, m_s\}$, we get the $\mathbb{C}$-algebra homomorphism

$$\pi : \mathbb{C}[x_1, \ldots, x_s] \longrightarrow \mathbb{C}[M]$$

where $x_i \mapsto \chi^{m_i} \in M$. This corresponds to the morphism

$$\Phi_{\mathscr{A}} : T_N \longrightarrow \mathbb{C}^s$$

from (1.4), i.e., $\pi = (\Phi_{\mathscr{A}})^*$ in the notation of §0. One checks that the kernel of π is the toric ideal $\mathbf{I}(Y_{\mathscr{A}})$ (Exercise 1.4). Since the image of π is $\mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}] = \mathbb{C}[\mathsf{S}]$, the coordinate ring of $Y_{\mathscr{A}}$ is

$$\begin{aligned} \mathbb{C}[Y_{\mathscr{A}}] &= \mathbb{C}[x_1, \ldots, x_n]/\mathbf{I}(Y_{\mathscr{A}}) \\ &= \mathbb{C}[x_1, \ldots, x_n]/\operatorname{Ker}(\pi) \simeq \operatorname{Im}(\pi) = \mathbb{C}[\mathsf{S}]. \end{aligned} \tag{1.5}$$

Thus $\mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ is the toric variety $Y_{\mathscr{A}}$. Since $\mathsf{S} = \mathbb{N}\mathscr{A}$ implies $\mathbb{Z}\mathsf{S} = \mathbb{Z}\mathscr{A}$, the torus of $Y_{\mathscr{A}} = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ has the desired character lattice by Proposition 1.8. $\square$

Here is an example of this proposition.

Example 1.15. Consider the affine semigroup $\mathsf{S} \subseteq \mathbb{Z}$ generated by 2 and 3, so that $\mathsf{S} = \{0, 2, 3, \dots\}$. To study the semigroup algebra $\mathbb{C}[\mathsf{S}]$, we use (1.5). If $\mathscr{A} = \{2, 3\}$, then $\Phi_{\mathscr{A}}(t) = (t^2, t^3)$ and the toric ideal is $\mathbf{I}(Y_{\mathscr{A}}) = \langle x^3 - y^2\rangle$ by Example 1.4. Hence

$$\mathbb{C}[\mathsf{S}] = \mathbb{C}[t^2, t^3] \simeq \mathbb{C}[x, y]/\langle x^3 - y^2\rangle$$

and the affine toric variety $Y_{\mathscr{A}}$ is the curve $x^3 = y^2$. $\diamond$

Equivalence of Constructions. We can now state the main result of this section, which asserts that our various approaches to affine toric varieties all give the same class of objects.

Theorem 1.16. *Let V be an affine variety. The following are equivalent:*

(a) *V is an affine toric variety according to Definition 1.3.*

(b) *$V = Y_{\mathscr{A}}$ for a finite set $\mathscr{A}$ in a lattice.*

(c) *V is an affine variety defined by a toric ideal.*

(d) *$V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S.*

Proof. The implications (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (a) follow from Propositions 1.8, 1.9 and 1.14. For (a) $\Rightarrow$ (d), let V be an affine toric variety containing the torus T_N, and let M be the character lattice of T_N. Since the coordinate ring of T_N is the semigroup algebra $\mathbb{C}[M]$, the inclusion $T_N \subseteq V$ induces a map of coordinate rings

$$\mathbb{C}[V] \longrightarrow \mathbb{C}[M].$$

This map is injective since T_N is Zariski dense in V, so that we can regard $\mathbb{C}[V]$ as a subalgebra of $\mathbb{C}[M]$. Let

$$\mathsf{S} = \{m \in M \mid \chi^m \in \mathbb{C}[V]\}$$

and note that S is a semigroup. We will show that S is finitely generated with semigroup algebra equal to $\mathbb{C}[V]$. This will complete the proof of the theorem.

The inclusion $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[V]$ is obvious. For the opposite inclusion, pick $f \neq 0$ in $\mathbb{C}[V]$. Using $\mathbb{C}[V] \subseteq \mathbb{C}[M]$, we can write

$$f = \sum_{m \in \mathscr{B}} c_m \chi^m,$$

where $\mathscr{B} \subseteq M$ is finite and $c_m \neq 0$ for all $m \in \mathscr{B}$. Let

$$W = \mathrm{Span}(\chi^m \mid m \in \mathscr{B}) \subseteq M_{\mathbb{R}}$$

be the subspace spanned by the χ^m. Thus $f \in W \cap \mathbb{C}[V]$. If $t \in T_N$, then the action of t on V induces an action on the coordinate ring $\mathbb{C}[V]$, so that we get

$t \cdot f \in \mathbb{C}[V]$. By Definition 1.3, this extends the usual action of t on T_N, which in terms of the coordinate ring $\mathbb{C}[M]$ is given by $t \cdot \chi^m = \chi^m(t)\chi^m$. It follows that W and hence $W \cap \mathbb{C}[V]$ are stable under the action of T_N. Since $W \cap \mathbb{C}[V]$ is finite-dimensional, Proposition 1.2 implies that $W \cap \mathbb{C}[V]$ is spanned by simultaneous eigenvectors of T_N. But this is taking place in $\mathbb{C}[M]$, where the simultaneous eigenvectors are the characters! It follows that $W \cap \mathbb{C}[V]$ is spanned by characters. Then the above expression for $f \in W \cap \mathbb{C}[V]$ implies that $\chi_m \in \mathbb{C}[V]$ for $m \in \mathscr{B}$. It follows that $f \in \mathbb{C}[\mathsf{S}]$, proving that $\mathbb{C}[V] = \mathbb{C}[\mathsf{S}]$.

It remains to show that S is finitely generated. Since $\mathbb{C}[V]$ is finitely generated, we can find $f_1, \ldots, f_s \in \mathbb{C}[V]$ with $\mathbb{C}[V] = \mathbb{C}[f_1, \ldots, f_s]$. Expressing each f_i in terms of characters as above gives the desired finite generating set of S. Hence S is an affine semigroup. $\square$

Here is one way to think about the above proof. When an irreducible affine variety V contains a torus T_N as a Zariski open subset, we have the inclusion

$$\mathbb{C}[V] \subseteq \mathbb{C}[M].$$

Thus $\mathbb{C}[V]$ consists of those functions on the torus T_N that extend to polynomial functions on V. Then the key insight is that V is a toric variety *precisely when the functions that extend are determined by the characters that extend.*

Example 1.17. We've seen that $V = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is a toric variety with toric ideal $\langle xy - zw\rangle \subseteq \mathbb{C}[x, y, z, w]$. Also, the torus is $(\mathbb{C}^*)^3$ via the map $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1t_2t_3^{-1})$. The lattice points used in this map can be represented as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{1.6}$$

The corresponding semigroup $\mathsf{S} \subseteq \mathbb{Z}^3$ consists of the $\mathbb{N}$-linear combinations of the column vectors. Hence the elements of S are lattice points lying in the polyhedral region in $\mathbb{R}^3$ pictured in Figure 1. In this figure, the four vectors generating S are shown in bold, and the boundary of the polyhedral region is partially shaded. In the terminology of §2, this polyhedral region is a *rational polyhedral cone*. In Exercise 1.5 you will show that S consists of *all* lattice points lying in the cone in Figure 1. We will use this in §3 to prove that V is normal. $\Diamond$

Exercises for §1.

1.1. As in Example 1.6, let

$$I = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1 \rangle \subseteq \mathbb{C}[x_0, \ldots, x_d]$$

and let $\widehat{C}_d$ be the surface parametrized by

$$\Phi(s, t) = (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d) \in \mathbb{C}^{d+1}.$$

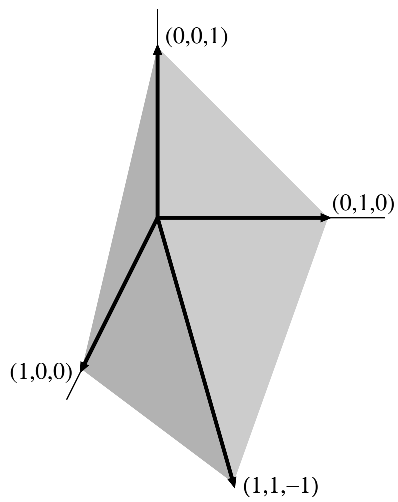


Figure 1. Cone containing the lattice points corresponding to $V = \mathbf{V}(xy - zw)$

(a) Prove that $\mathbf{V}(I) = \Phi(\mathbb{C}^2) \subseteq \mathbb{C}^{d+1}$. Thus $\widehat{C}_d = \mathbf{V}(I)$.

(b) Prove that $\mathbf{I}(\widehat{C}_d)$ is homogeneous.

(c) Consider lex monomial order with $x_0 > x_1 > \cdots > x_d$. Let $f \in \mathbf{I}(\widehat{C}_d)$ be homogeneous of degree ℓ and let r be the remainder of f on division by the generators of I. Prove that r can be written

$$r = h_0(x_0, x_1) + h_1(x_1, x_2) + \cdots + h_{d-1}(x_{d-1}, x_d)$$

where h_i is homogeneous of degree ℓ. Also explain why we may assume that the coefficient of x_i^ℓ in h_i is zero for $1 \le i \le d-1$.

(d) Use part (c) and $r(s^d, s^{d-1}t, \ldots, st^{d-1}, t^d) = 0$ to show that $r = 0$.

(e) Use parts (b), (c) and (d) to prove that $I = \mathbf{I}(\widehat{C}_d)$. Also explain why the generators of I are a Gröbner basis for the above lex order.

1.2. Let $L \subseteq \mathbb{Z}^s$ be a sublattice. Prove that

$$\langle x^{\ell_+} - x^{\ell_-} \mid \ell \in L\rangle = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s,\ \alpha - \beta \in L\rangle.$$

Note that when $\ell \in L$, the vectors $\ell_+, \ell_- \in \mathbb{N}^s$ have disjoint support (i.e., no coordinate is positive in both), while this may fail for arbitrary $\alpha, \beta \in \mathbb{N}^s$ with $\alpha - \beta \in L$.

1.3. Let I_L be a toric ideal and let $\ell^1, \ldots, \ell^r$ be a basis of the sublattice $L \subseteq \mathbb{Z}^s$. Define

$$I_{\mathbf{L}} = \langle x^{\ell^i_+} - x^{\ell^i_-} \mid i = 1, \ldots, r\rangle.$$

Prove that $I_L = I_{\mathbf{L}} : \langle x_1 \cdots x_s\rangle^\infty$. Hint: Given $\alpha, \beta \in \mathbb{N}^s$ with $\alpha - \beta \in L$, write $\alpha - \beta = \sum_{i=1}^r a_i \ell^i$, $a_i \in \mathbb{Z}$. This implies

$$x^{\alpha-\beta} - 1 = \prod_{a_i > 0} \left(\frac{x^{\ell^i_+}}{x^{\ell^i_-}}\right)^{a_i} \prod_{a_i < 0} \left(\frac{x^{\ell^i_-}}{x^{\ell^i_+}}\right)^{-a_i} - 1.$$

Show that multiplying this by $(x_1 \cdots x_s)^k$ gives an element of $I_{\mathbf{L}}$ for $k \gg 0$. (By being more careful, one can show that this result holds for lattice ideals. See [**17**, Lemma 7.6].)

1.4. Fix an affine variety V and $f_1, \ldots, f_s \in \mathbb{C}[V]$. This gives a polynomial map $\Phi : V \to \mathbb{C}^s$, which on coordinate rings is given by

$$\Phi^* : \mathbb{C}[x_1, \ldots, x_s] \longrightarrow \mathbb{C}[V], \qquad x_i \longmapsto f_i.$$

Let $Y \subseteq \mathbb{C}^s$ be the Zariski closure of the image of Φ.

(a) Prove that $\mathbf{I}(Y) = \mathrm{Ker}(\Phi^*)$.

(b) Explain how this applies to the proof of Proposition 1.14.

1.5. Let $m_1 = (1,0,0), m_2 = (0,1,0), m_3 = (0,0,1), m_4 = (1,1,-1)$ be the columns of the matrix in Example 1.17 and let

$$C = \Big\{ \sum_{i=1}^{4} \lambda_i m_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \Big\} \subseteq \mathbb{R}^3$$

be the cone in Figure 1. Prove that $C \cap \mathbb{Z}^3$ is a semigroup generated by m_1, m_2, m_3, m_4.

1.6. An interesting observation is that different sets of lattice points can parametrize the same affine toric variety, even though these parametrizations behave slightly differently. In this exercise you will consider the parametrizations

$$\Phi_1(s,t) = (s^2, st, st^3) \quad \text{and} \quad \Phi_2(s,t) = (s^3, st, t^3).$$

(a) Prove that Φ_1 and Φ_2 both give the affine toric variety $Y = \mathbf{V}(xz - y^3) \subseteq \mathbb{C}^3$.

(b) We can regard Φ_1 and Φ_2 as maps

$$\Phi_1 : \mathbb{C}^2 \longrightarrow Y \quad \text{and} \quad \Phi_2 : \mathbb{C}^2 \longrightarrow Y.$$

Prove that Φ_2 is surjective and that Φ_1 is not.

In general, a finite subset $\mathscr{A} \subseteq \mathbb{Z}^n$ gives a rational map $\Phi_{\mathscr{A}} : \mathbb{C}^n \dashrightarrow Y_{\mathscr{A}}$. The image of $\Phi_{\mathscr{A}}$ in $\mathbb{C}^s$ is called a *toric set* in the literature. Thus $\Phi_1(\mathbb{C}^2)$ and $\Phi_2(\mathbb{C}^2)$ are toric sets. The papers [**16**] and [**21**] study when a toric set equals the corresponding affine toric variety.

1.7. In Example 1.6 and Exercise 1.1 we constructed the rational normal cone $\widehat{C}_d$ using all monomials of degree d in s, t. If we drop some of the monomials, things become more complicated. For example, consider the surface parametrized by

$$\Phi(s,t) = (s^4, s^3t, st^3, t^4) \in \mathbb{C}^4.$$

This gives a toric variety $Y \subseteq \mathbb{C}^4$. Show that the toric ideal of Y is given by

$$\mathbf{I}(Y) = \langle xw - yz, yw^2 - z^3, xz^2 - y^2w, x^2z - y^3 \rangle \subseteq \mathbb{C}[x,y,z,w].$$

The toric ideal for $\widehat{C}_4$ has quadratic generators; by removing the monomial s^2t^2, we now get cubic generators. In Chapter 2 we will use this example to construct a projective curve that is normal but not projectively normal.

1.8. Prove that a lattice ideal I_L for $L \subseteq \mathbb{Z}^s$ is a toric ideal if and only if $\mathbb{Z}^s/L$ is torsion-free. Hint: When $\mathbb{Z}^s/L$ is torsion-free, it can be regarded as the character lattice of a torus. The other direction of the proof is more challenging. If you get stuck, see [**17**, Theorem 7.4].

1.9. Prove that $I = \langle x^2 - 1, xy - 1, yz - 1\rangle$ is the lattice ideal for the lattice

$$L = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c \equiv 0 \bmod 2\} \subseteq \mathbb{Z}^3.$$

Also compute primary decomposition of I to show that I is not prime.

1.10. Let T_N be a torus with character lattice M. Then every point $t \in T_N$ gives an evaluation map $\phi_t : M \to \mathbb{C}^*$ defined by $\phi_t(m) = \chi^m(t)$. Prove that ϕ_t is a group homomorphism and that the map $t \mapsto \phi_t$ induces a group isomorphism

$$T_N \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

1.11. Consider tori T_1 and T_2 with character lattices M_1 and M_2. By Example 1.13, the coordinate rings of T_1 and T_2 are $\mathbb{C}[M_1]$ and $\mathbb{C}[M_2]$. Let $\Phi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then Φ induces maps

$$\widehat{\Phi} : M_2 \longrightarrow M_1 \quad \text{and} \quad \Phi^* : \mathbb{C}[M_2] \longrightarrow \mathbb{C}[M_1]$$

by composition. Prove that Φ^* is the map of semigroup algebras induced by the map $\widehat{\Phi}$ of affine semigroups.

1.12. A commutative semigroup S is *cancellative* if $u + v = u + w$ implies $v = w$ for all $u, v, w \in \mathsf{S}$ and *torsion-free* if $nu = 0$ implies $u = 0$ for all $n \in \mathbb{N} \setminus \{0\}$ and $u \in \mathsf{S}$. Prove that S is affine if and only if it is finitely generated, cancellative, and torsion-free.

1.13. The requirement that an affine semigroup be finitely generated is important since lattices contain semigroups that are not finitely generated. For example, let $\tau > 0$ be irrational and consider the semigroup

$$\mathsf{S} = \{(a,b) \in \mathbb{N}^2 \mid b \geq \tau a\} \subseteq \mathbb{Z}^2.$$

Prove that S is not finitely generated. (When τ satisfies a quadratic equation with integer coefficients, the generators of S are related to continued fractions. For example, when $\tau = (1+\sqrt{5})/2$ is the golden ratio, the minimal generators of S are $(1,0)$ and (F_{2n-1}, F_{2n}) for $n = 1, 2, \ldots$, where F_n is the nth Fibonacci number. See [**25**] for further details.)

1.14. Suppose that $\phi : M \to M$ is a group isomorphism. Fix a finite set $\mathscr{A} \subseteq M$ and let $\mathscr{B} = \phi(\mathscr{A})$. Prove that the toric varieties $Y_{\mathscr{A}}$ and $Y_{\mathscr{B}}$ are equivariantly isomorphic (meaning that the isomorphism respects the torus action).

2. Cones and Affine Toric Varieties

We begin with a brief discussion of rational polyhedral cones and then explain how they relate to affine toric varieties.

Polyhedral Cones. Fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ of dimension n. Our discussion will omit most proofs—we refer the reader to [**10**] for a more detailed treatment and [**20**, Appendix A.1] for careful statements. See also [**3, 11, 22**].

Definition 2.1. A ***polyhedral cone*** in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \Big\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \Big\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say that σ is ***generated*** by S. Also set $\mathrm{Cone}(\emptyset) = \{0\}$.

One easily checks that polyhedral cone σ is *convex*, meaning that $x, y \in \sigma \Rightarrow \lambda x + (1-\lambda)y \in \sigma$ for all $0 \leq \lambda \leq 1$, and is a *cone*, meaning $x \in \sigma \Rightarrow \lambda x \in \sigma$ for all $\lambda \geq 0$. Cones satisfying Definition 2.1 are sometimes called "convex polyhedral cones", but we will drop the "convex" since all cones we consider will be convex.

Examples of convex polyhedral cones include the first quadrant in $\mathbb{R}^2$ or first octant in $\mathbb{R}^3$. For another example, the cone $\mathrm{Cone}(e_1, e_2, e_1 + e_3.e_2 + e_3) \subseteq \mathbb{R}^3$ is pictured in Figure 2. It is also possible to have cones that contain entire lines.

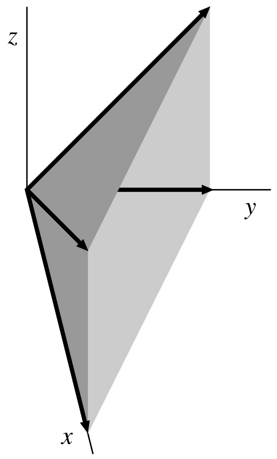


Figure 2. Cone in $\mathbb{R}^3$ generated by $e_1, e_2, e_1 + e_3.e_2 + e_3$

For example, $\mathrm{Cone}(e_1, -e_1) \subseteq \mathbb{R}^2$ is the x-axis, while $\mathrm{Cone}(e_1, -e_1, e_2)$ is the closed upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. As we will see below, these last two examples are not *strongly convex*.

We can also create cones using *polytopes*, which are defined as follows.

Definition 2.2. A ***polytope*** in $N_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \Big\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0,\ \sum_{u \in S} \lambda_u = 1 \Big\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say that P is the ***convex hull*** of S.

Polytopes include all polygons in $\mathbb{R}^2$ and polyhedra in $\mathbb{R}^3$. As we will see in later chapters, polytopes play a prominent role in the theory of toric varieties. Here, however, we simply observe that a polytope $P \subseteq N_{\mathbb{R}}$ gives a polyhedral cone in $N_{\mathbb{R}} \times \mathbb{R}$ by taking the cone

$$\sigma = \{\lambda \cdot (u, 1) \in N_{\mathbb{R}} \times \mathbb{R} \mid u \in P,\ \lambda \geq 0\}.$$

If $P = \mathrm{Conv}(S)$, then we can also describe this as $\sigma = \mathrm{Cone}(S \times \{1\})$. Figure 3 on the next page shows what this looks when P is a pentagon in the plane

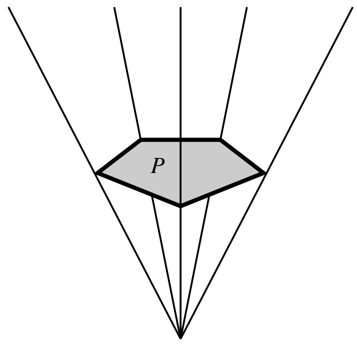


Figure 3. Cone over a pentagon $P \subseteq \mathbb{R}^2$

The *dimension* of a polyhedral cone σ, denoted $\dim \sigma$, is the dimension of the smallest subspace W of $N_{\mathbb{R}}$ containing σ. We call W the *span* of σ.

Dual Cones and Faces. As usual, the pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ is denoted $\langle\ ,\ \rangle$.

Definition 2.3. Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its ***dual cone*** is defined by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Duality has the following important properties.

Proposition 2.4. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then $\sigma^\vee$ is a polyhedral cone in $M_{\mathbb{R}}$ and $(\sigma^\vee)^\vee = \sigma$.* □

Given $m \neq 0$ in $N_{\mathbb{R}}$, we get the hyperplane

$$H_m = \{u \in M_{\mathbb{R}} \mid \langle m, u\rangle = 0\}$$

and the closed half-space

$$H_m^+ = \{u \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq 0\}.$$

We say that H_m is a *supporting hyperplane* of a polyhedral cone σ if $\sigma \subseteq H_m^+$, and H_m^+ is a *supporting half-space*. Note that H_m is a supporting hyperplane of σ if and only if $m \in \sigma^\vee \setminus \{0\}$. Furthermore, if $m_1, \ldots, m_s$ generate $\sigma^\vee$, then it is straightforward to check that

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_1}^+. \tag{2.1}$$

Thus every polyhedral cone is an intersection of finitely many closed half-spaces.

We can use supporting hyperplanes and half-spaces to define *faces* of a cone.

Definition 2.5. A ***face*** of the polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$. Using $m = 0$ shows that σ as a face of itself. Faces $\tau \neq \sigma$ are called ***proper faces***.

The faces of a polyhedral cone have the following obvious properties.

Lemma 2.6. *Let $\sigma = \mathrm{Cone}(S)$ be a polyhedral cone. Then:*

(a) *Every face of σ is a polyhedral cone.*

(b) *An intersection of two faces of σ is again a face of σ.*

(c) *A face of a face of σ is again a face of σ.* $\square$

A *facet* of σ is a face τ of codimension 1, i.e., $\dim \tau = \dim \sigma - 1$. An *edge* of σ is a face of dimension 1. In Figure 4 we illustrate a 3-dimensional cone with shaded facets and a supporting hyperplane (a plane in this case) that cuts out the vertical edge of the cone.

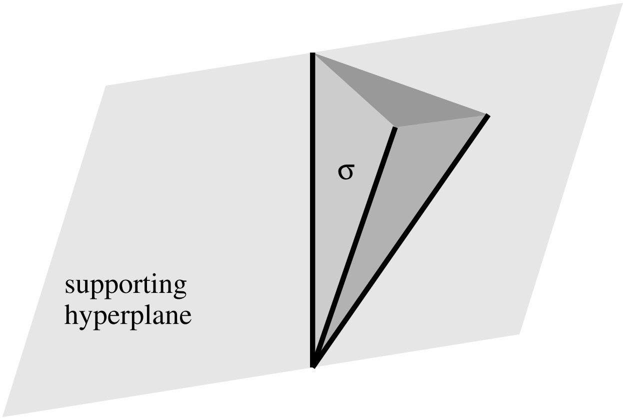


Figure 4. A cone $\sigma \subseteq \mathbb{R}^3$ with shaded facets and a supporting hyperplane

Here are some properties of facets.

Proposition 2.7. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a polyhedral cone. Then:*

(a) *If $\dim \sigma = n$ and the facets of σ are $\tau_i = H_{m_i} \cap \sigma$ for $m_i \in \sigma^\vee$, $1 \leq i \leq s$, then*

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+ \quad \text{and} \quad \sigma^\vee = \mathrm{Cone}(m_1, \ldots, m_s).$$

(b) *Every proper face τ of σ is the intersection of the facets of σ containing τ.* $\square$

Note how part (a) of the proposition refines (2.1) when $\dim \sigma = \dim N_{\mathbb{R}}$.

When working in $\mathbb{R}^n$, dot product allows us to identify the dual with $\mathbb{R}^n$. From this point of view, the vectors $m_1, \ldots, m_s$ in part (a) of the proposition are *facet normals*, i.e., perpendicular to the facets. This makes it easy to compute examples.

Example 2.8. It easy to see that the facet normals to the cone $\sigma \subseteq \mathbb{R}^3$ in Figure 2 are $m_1 = e_1, m_2 = e_2, m_3 = e_3, m_4 = e_1 + e_2 - e_3$. Hence

$$\sigma^\vee = \mathrm{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3) \subseteq \mathbb{R}^3.$$

This is the cone of Figure 1 at the end of §1 whose lattice points describe the semigroup of the affine toric variety $\mathbf{V}(xy - zw)$ (see Example 1.17). As we will see, this is part of how cones describe normal affine toric varieties.

Now consider $\sigma^\vee$, which is the cone in Figure 1. The reader can check that the facet normals of this cone are $e_1, e_2, e_1 + e_3, e_2 + e_3$. Using duality and part (b) of Proposition 2.7, we obtain

$$\sigma = (\sigma^\vee)^\vee = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3).$$

Hence we recover our original description of σ. ◇

In this example, facets of the cone correspond to edges of its dual. More generally, given a face τ of a polyhedral cone $\sigma \subseteq N_\mathbb{R}$, we define

$$\begin{aligned} \tau^\perp &= \{m \in M_\mathbb{R} \mid \langle m, u\rangle = 0 \text{ for all } u \in \tau\} \\ \tau^* &= \{m \in \sigma^\vee \mid \langle m, u\rangle = 0 \text{ for all } u \in \tau\} \\ &= \sigma^\vee \cap \tau^\perp. \end{aligned}$$

We call τ^* the *dual face* of τ because of the following proposition.

Proposition 2.9. *If τ is a face of a polyhedral cone σ and $\tau^* = \sigma^\vee \cap \tau^\perp$, then:*

(a) *τ^* is a face of $\sigma^\vee$.*

(b) *The map $\tau \mapsto \tau^*$ is a bijective inclusion-reversing correspondance between the faces of σ and the faces of $\sigma^\vee$.*

(c) $\dim \tau + \dim \tau^* = n$. □

Here is an example of Proposition 2.9 when $\dim \sigma < \dim N_\mathbb{R}$.

Example 2.10. Let $\sigma = \mathrm{Cone}(e_1, e_2) \subseteq \mathbb{R}^3$. Figure 5 shows σ and $\sigma^\vee$. You

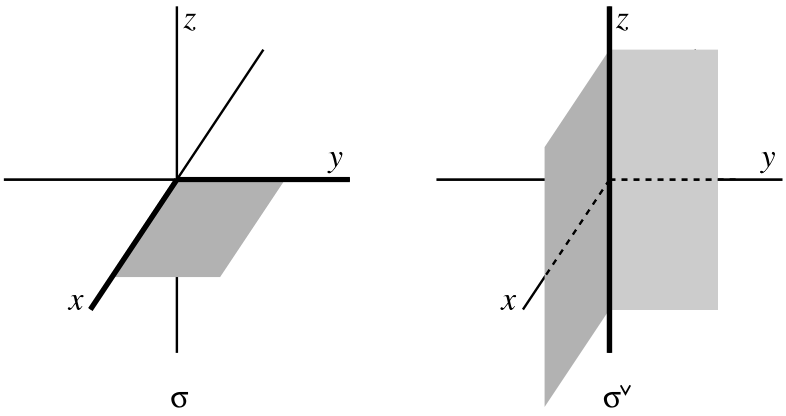


Figure 5. A 2-dimensional cone $\sigma \subseteq \mathbb{R}^3$ and its dual $\sigma^\vee \subseteq \mathbb{R}^3$

should check that the maximal face of σ, namely σ itself, gives the minimal face σ^* of $\sigma^\vee$, namely the z-axis. Note also that

$$\dim \sigma + \dim \sigma^* = 3$$

even though σ has dimension 2. $\Diamond$

Strong Convexity. Of the cones shown in Figures 1–5, all but $\sigma^\vee$ in Figure 5 have the nice property that the origin is a face. Such cones are called *strongly convex*. This condition can be stated several ways.

Proposition 2.11. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a polyhedral cone. Then:*

$$\begin{aligned} \sigma \text{ is strongly convex} &\iff \{0\} \text{ is a face of } \sigma \\ &\iff \sigma \text{ contains no positive-dimensional subspace} \\ &\iff \sigma \cap (-\sigma) = \{0\} \\ &\iff \dim \sigma^\vee = n. \end{aligned}$$ $\square$

You will prove Proposition 2.11 in Exercise 2.1. One corollary is that if a polyhedral cone σ is strongly convex of maximal dimension, then so is $\sigma^\vee$. The cones pictured in Figures 1–4 satisfy this condition.

In general, a polyhedral cone σ always has a minimal face that is the largest subspace W contained in σ. Furthermore, we have $W = \sigma \cap (-\sigma) = (\sigma^\vee)^\perp$, and $\overline{\sigma} = \sigma/W \subseteq N_{\mathbb{R}}/W$ is a strongly convex polyhedral cone. See Exercise 2.2.

Rational Polyhedral Cones. Let N and M be dual lattices with associated vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For $\mathbb{R}^n$ we usually use the lattice $\mathbb{Z}^n$.

Definition 2.12. A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is ***rational*** if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$.

The cones appearing in Figures 1, 2 and 5 are rational. We note without proof that faces and duals of rational polyhedral cones are rational. Furthermore, if $\sigma = \mathrm{Cone}(S)$ for $S \subseteq N$ finite and $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, then

$$\sigma \cap N_{\mathbb{Q}} = \big\{ \textstyle\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \text{ in } \mathbb{Q} \big\}. \tag{2.2}$$

One new feature is that a strongly convex rational polyedral cone σ has a canonical generating set, constructed as follows. Let ρ be a edge of σ. Since σ is strongly convex, ρ is a *ray*, i.e., a half-line, and since ρ is rational, the semigroup $\rho \cap N$ is generated by a unique element u of the intersection. We call u the *ray generator* of ρ. Figure 6 on the next page shows the ray generator of a rational ray ρ in the plane. The dots are the lattice $N = \mathbb{Z}^2$ and the white ones are $\rho \cap N$.

Lemma 2.13. *A strongly convex rational polyedral cone is generated by the ray generators of its edges.* $\square$

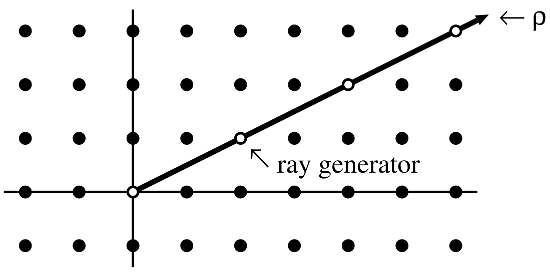


Figure 6. A rational ray $\rho \subseteq \mathbb{R}^2$ and its unique ray generator

It is customary to call the ray generators of the edges the *minimal generators* of a strongly convex rational polyhedral cone. Figures 1 and 2 show 3-dimensional strongly convex rational polyhedral cones and their ray generators.

In a similar way, a rational polyhedral cone σ of maximal dimension has unique *facet normals*, which are the ray generators of the dual $\sigma^\vee$, which is strongly convex by Proposition 2.11.

Here are some especially nice strongly convex cones that play an important role in toric geometry.

Definition 2.14. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone.

(a) σ is ***smooth*** or ***regular*** if its minimal generators form part of a $\mathbb{Z}$-basis of N,

(b) σ is ***simplicial*** if its minimal generators are linearly independent over $\mathbb{R}$.

The cones pictured in Figure 5 are smooth, while those in Figures 1 and 2 are not even simplicial. Note also that the dual of a smooth resp. simplicial cone is again smooth resp. simplicial. Later in the section we will give examples of simplicial cones that are not smooth.

Semigroup Algebras and Affine Toric Varieties. Given a rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$, the lattice points

$$\mathsf{S}_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup. A key fact is that this semigroup is finitely generated.

Proposition 2.15 (Gordan's Lemma)**.** $\mathsf{S}_\sigma = \sigma^\vee \cap M$ *is finitely generated and hence is an affine semigroup.*

Proof. Since $\sigma^\vee$ is rational polyhedral, $\sigma^\vee = \mathrm{Cone}(T)$ for a finite set $T \subseteq M$. Then $K = \{\sum_{m\in T} \delta_m m \mid 0 \leq \delta_u < 1\}$ is a bounded region of $M_\mathbb{R} \simeq \mathbb{R}^n$, so that $K \cap M$ is finite since $M \simeq \mathbb{Z}^n$. Note that $T \cup (K \cap M) \subseteq \mathsf{S}_\sigma$.

We claim $T \cup (K \cap M)$ generates S_σ as a semigroup. To prove this, take $w \in \mathsf{S}_\sigma$ and write $w = \sum_{m\in T} \lambda_m m$ where $\lambda_m \geq 0$. Then $\lambda_m = \lfloor \lambda_m \rfloor + \delta_m$ with

$\lfloor \lambda_m \rfloor \in \mathbb{N}$ and $0 \le \delta_m < 1$, so that

$$w = \sum_{m \in T} \lfloor \lambda_m \rfloor m + \sum_{m \in T} \delta_m m.$$

The second sum is in $K \cap M$ (remember $w \in M$). It follows that w is a nonnegative integer combination of elements of $T \cup (K \cap M)$. □

From §1, we know that affine semigroups give affine toric varieties. Hence we get the following result.

Theorem 2.16. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational polyhedral cone with semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Then*

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_\sigma]) = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety. Furthermore,

$$\dim U_\sigma = n \iff \text{the torus of } U_\sigma \text{ is } T_N \iff \sigma \text{ is strongly convex}.$$

Proof. By Gordan's Lemma and Proposition 1.14, U_σ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathsf{S}_\sigma \subseteq M$. To study $\mathbb{Z}\mathsf{S}_\sigma$, note that

$$\mathbb{Z}\mathsf{S}_\sigma = \mathsf{S}_\sigma - \mathsf{S}_\sigma = \{m_1 - m_2 \mid m_1, m_2 \in \mathsf{S}_\sigma\}.$$

Now suppose that $km \in \mathbb{Z}\mathsf{S}_\sigma$ for some $k > 1$ and $m \in M$. Then $km = m_1 - m_2$ for $m_1, m_2 \in \mathsf{S}_\sigma = \sigma^\vee \cap M$. Since m_1 and m_2 lie in the convex set $\sigma^\vee$, we have

$$m + m_2 = \tfrac{1}{k} m_1 + \tfrac{k-1}{k} m_2 \in \sigma^\vee.$$

It follows that $m = (m + m_2) - m_2 \in \mathbb{Z}\mathsf{S}_\sigma$, so that $M/\mathbb{Z}\mathsf{S}_\sigma$ is torsion-free. Hence

$$\text{(2.3)} \qquad \text{the torus of } U_\sigma \text{ is } T_N \iff \mathbb{Z}\mathsf{S}_\sigma = M \iff \operatorname{rank} \mathbb{Z}\mathsf{S}_\sigma = n.$$

Since σ is strongly convex if and only if $\dim \sigma^\vee = n$ (Proposition 2.11), it remains to show that

$$\dim U_\sigma = n \iff \operatorname{rank} \mathbb{Z}\mathsf{S}_\sigma = n \iff \dim \sigma^\vee = n.$$

The first equivalence follows since the dimension of an affine toric variety is the dimension of its torus, which is the rank of its character lattice. We leave the proof of the second equivalence to the reader (Exercise 2.3). □

Since we want our affine toric varieties to contain the torus T_N, we consider only those affine toric varieties U_σ for which $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex.

Our first example of Theorem 2.16 is an affine toric variety we know well.

Example 2.17. Let $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq N_{\mathbb{R}} = \mathbb{R}^3$ with $N = \mathbb{Z}^3$. This is the cone pictured in Figure 2. By Example 2.8, $\sigma^\vee$ is the cone pictured in Figure 1, and by Example 1.17, the lattice points in this cone are generated by columns of matrix (1.6). It follows from Example 1.17 that U_σ is the affine toric variety $\mathbf{V}(xy - zw)$. ◇

Here are two further examples of Theorem 2.16.

Example 2.18. Fix $0 \leq r \leq n$ and set $\sigma = \mathrm{Cone}(e_1, \ldots, e_r) \subseteq \mathbb{R}^n$. Then

$$\sigma^\vee = \mathrm{Cone}(e_1, \ldots, e_r, \pm e_{r+1}, \ldots, \pm e_n)$$

and the corresponding affine toric variety is

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]) = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$$

(Exercise 2.4). This implies the general fact that if $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^n$ is a smooth cone of dimension r, then $U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$. $\diamond$

Figure 5 illustrates the cones involved in Example 2.18 when $r = 2$ and $n = 3$.

Example 2.19. Fix a positive integer d and let $\sigma = \mathrm{Cone}(e_1, de_1 - e_2) \subseteq \mathbb{R}^2$. This has dual cone $\sigma^\vee = \mathrm{Cone}(e_1, e_1 + de_2)$. Figure 7 shows $\sigma^\vee$ when $d = 4$. The affine semigroup $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ is generated by the lattice points $(1, i)$ for $0 \leq i \leq d$. When $d = 4$, these are the white dots in Figure 7. (You will prove these assertions in Exercise 2.5.)

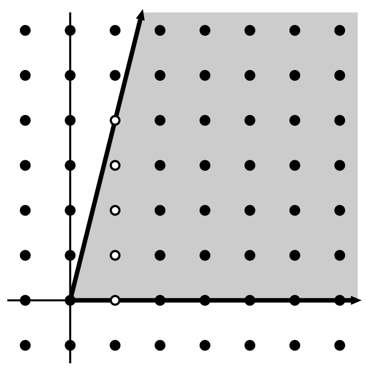


Figure 7. The cone $\sigma^\vee$ when $d = 4$

By §1, the affine toric variety U_σ is the Zariski closure of the image of the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{C}^{d+1}$ defined by

$$\Phi(s, t) = (s, st, st^2, \ldots, st^d).$$

This map has the same image as the map $(s, t) \mapsto (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)$ used in Example 1.6. Thus U_σ is isomorphic to the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ whose ideal is generated by the 2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix}.$$

Note that the cones σ and $\sigma^\vee$ are simplicial but not smooth. $\diamond$

We will return to this example often as we develop the theory of toric varieties. One thing evident in Example 1.6 is the difference between *cone generators* and *semigroup generators*: the cone $\sigma^\vee$ has two generators but the semigroup $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ has $d+1$.

When $\sigma \subseteq N_\mathbb{R}$ has maximal dimension, the semigroup $S_\sigma = \sigma^\vee \cap M$ has a unique minimal generating set constructed as follows. Define an element $m \neq 0$ of S_σ to be *irreducible* if $m = m' + m''$ for $m', m'' \in \mathsf{S}_\sigma$ implies $m' = 0$ or $m'' = 0$.

Proposition 2.20. *Let $\sigma \subseteq N_\mathbb{R}$ be strongly convex of maximal dimension and let $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Then*

$$\mathscr{H} = \{m \in \mathsf{S}_\sigma \mid m \textit{ is irreducible}\}$$

has the following properties:

(a) *$\mathscr{H}$ is finite and generates S_σ.*

(b) *$\mathscr{H}$ contains the ray generators of the edges of $\sigma^\vee$.*

(c) *$\mathscr{H}$ is the minimal generating set of S_σ with respect to inclusion.*

Proof. Proposition 2.11 implies that $\sigma^\vee$ is strongly convex, so we can find an element $u \in \sigma \cap N \setminus \{0\}$ such that $\langle m, u\rangle \in \mathbb{N}$ for all $m \in \mathsf{S}_\sigma$ and $\langle m, u\rangle = 0$ if and only if $m = 0$.

Now suppose that $m \in \mathsf{S}$ is not irreducible. Then $m = m' + m''$ where m' and m'' are nonzero elements of S_σ. It follows that

$$\langle m, u\rangle = \langle m', u\rangle + \langle m'', u\rangle$$

with $\langle m', u\rangle, \langle m'', u\rangle \in \mathbb{N} \setminus \{0\}$, so that

$$\langle m', u\rangle < \langle m, u\rangle \quad \text{and} \quad \langle m'', u\rangle < \langle m, u\rangle.$$

Using induction on $\langle m, u\rangle$, we conclude that every element of S_σ is a sum of irreducible elements, so that $\mathscr{H}$ generates S_σ. Furthermore, using a finite generating set of S_σ, one easily sees that $\mathscr{H}$ is finite. This proves part (a). The remaining parts of the proof are covered in Exercise 2.6. $\square$

The set $\mathscr{H} \subseteq \mathsf{S}_\sigma$ is called the *Hilbert basis* of S_σ and its elements are the *minimal generators* of S_σ. Algorithms for computing Hilbert bases are discussed in [**17**, Section 7.3] and [**26**, Chapter 13].

Exercises for §2.

2.1. Prove Proposition 2.11.

2.2. Let $\sigma \subseteq N_\mathbb{R}$ be a polyhedral cone.

(a) Use Proposition 2.9 to prove that σ has a unique minimal face with respect to inclusion. Let W denote this minimal face.

(b) Prove that $W = (\sigma^\vee)^\perp$.

(c) Prove that W is the largest subspace contained in σ.

(d) Prove that $W = \sigma \cap (-\sigma)$.

(e) Prove that $\overline{\sigma} = \sigma/W \subseteq N_{\mathbb{R}}/W$ is a strongly convex polyhedral cone.

2.3. Fix a lattice $M \simeq \mathbb{R}^n$ and let $\mathrm{Span}(S)$ denote the span over $\mathbb{R}$ of a subset $S \subseteq M_{\mathbb{R}}$.

(a) Let $S \subseteq M$ be finite. Prove that $\operatorname{rank} \mathbb{Z}S = \dim \mathrm{Span}(S)$.

(b) Let $S \subseteq M_{\mathbb{R}}$ be finite. Prove that $\dim \mathrm{Cone}(S) = \dim \mathrm{Span}(S)$.

(c) Use parts (a) and (b) to complete the proof of Theorem 2.16.

2.4. Prove the assertions made in Example 2.18.

2.5. Prove the assertions made in Example 2.19. Hint: First show that when a cone is smooth, the ray generators of the cone also generate the corresponding semigroup. Then write the cone $\sigma^\vee$ of Example 2.19 as a union of such cones.

2.6. Complete the proof of Proposition 2.20. Hint for part (b): Show that the ray generators of the edges of $\sigma^\vee$ are irreducible in S_σ. Given an edge ρ of $\sigma^\vee$, it will help to pick $u \in \sigma \cap N \setminus \{0\}$ such that $\rho = H_u \cap \sigma^\vee$.

2.7. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone generated by a set of linearly independent vectors in $N_{\mathbb{R}}$. Show that σ is strongly convex and simplicial.

2.8. Explain the picture illustrated in Figure 8 in terms of Proposition 2.7.

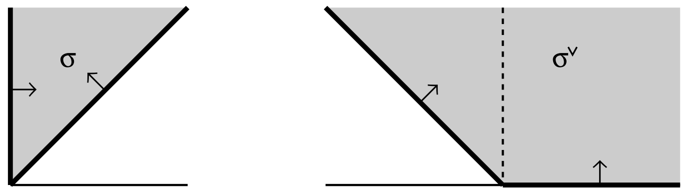


Figure 8. A cone σ in the plane and its dual

2.9. Let $P \subseteq N_{\mathbb{R}}$ be a polytope lying in an affine hyperplane (= translate of a hyperplane) not containing the origin. Generalize Figure 3 by showing that P gives a convex polyhedral cone in $N_{\mathbb{R}}$. Draw a picture.

2.10. Consider the cone $\sigma = \mathrm{Cone}(3e_1 - 2e_2, e_1) \subseteq \mathbb{R}^2$.

(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^2$. Draw a picture similar to Figure 7.

(b) Compute the toric ideal of the affine toric variety U_σ and explain how this exercise relates to Exercise 1.6.

2.11. Consider the simplicial cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_2 + 2e_3) \subseteq \mathbb{R}^3$.

(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^3$.

(b) Compute the toric ideal of the affine toric variety U_σ.

2.12. Let σ be a strongly convex polyhedral cone of maximal dimension. Here is an example taken from [**10**, p. 132] to show that σ and $\sigma^\vee$ need not have the same number of edges. Let $\sigma \subseteq \mathbb{R}^4$ be the cone generated by $2e_i + e_j$ for all $1 \le i, j \le 4$, $i \ne j$.

(a) Show that σ has 12 edges.

(b) Show that $\sigma^\vee$ is generated by e_i and $-e_i + 2\sum_{j \ne i} e_j$, $1 \le i \le 4$ and has 8 edges.

3. Properties of Affine Toric Varieties

The final task of this chapter is to explore the properties of affine toric varieties. We will also study maps between affine toric varieties.

Points of Affine Toric Varieties. We first consider various ways to describe the points of an affine toric variety.

Proposition 3.1. *Let* $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ *be the affine toric variety of the affine semigroup* S*. Then there are bijective correspondences between the following objects:*

(a) *Points* $p \in V$.

(b) *Maximal ideals* $\mathfrak{m} \subseteq \mathbb{C}[\mathsf{S}]$.

(c) *Semigroup homomorphisms* $\mathsf{S} \to \mathbb{C}$*, where* $\mathbb{C}$ *is considered as a semigroup under multiplication.*

Proof. The correspondence between (a) and (b) is standard (see [**6**, Theorem 5 of Chapter 5]). The correspondence between (a) and (c) is special to the toric case.

Given a point $p \in V$, define $\mathsf{S} \to \mathbb{C}$ by sending $m \in \mathsf{S}$ to $\chi^m(p) \in \mathbb{C}$. This makes sense since $\chi^m \in \mathbb{C}[\mathsf{S}] = \mathbb{C}[V]$. One easily checks that $\mathsf{S} \to \mathbb{C}$ is a semigroup homomorphism.

Going the other way, let $\gamma : \mathsf{S} \to \mathbb{C}$ be a semigroup homomorphism. Since $\{\chi^m\}_{m\in\mathsf{S}}$ is a basis of $\mathbb{C}[\mathsf{S}]$, γ induces a surjective linear map $\mathbb{C}[\mathsf{S}] \to \mathbb{C}$ which is a $\mathbb{C}$-algebra homomorphism. The kernel is a maximal ideal and thus gives a point $p \in V$ by the correspondence between (a) and (b).

We construct p concretely as follows. Let $\mathscr{A} = \{m_1, \ldots, m_s\}$ generate S, so that $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$. Let $p = (\gamma(m_1), \ldots, \gamma(m_s)) \in \mathbb{C}^s$. Let us prove that $p \in V$. By Proposition 1.9, it suffices to show that $x^\alpha - x^\beta$ vanishes at p for all exponent vectors $\alpha = (a_1, \ldots, a_s)$ and $\beta = (a_1, \ldots, a_s)$ satisfying

$$\sum_{i=1}^s a_i m_i = \sum_{i=1}^s b_i m_i.$$

This is easy, since γ being a semigroup homomorphism implies that

$$\prod_{i=1}^s \gamma(m_i)^{a_i} = \gamma\Big(\sum_{i=1}^s a_i m_i\Big) = \gamma\Big(\sum_{i=1}^s b_i m_i\Big) = \prod_{i=1}^s \gamma(m_i)^{b_i}.$$

It is straightforward to show that this point of V agrees with the one constructed in the previous paragraph (Exercise 3.1). □

As an application of this result, we describe the torus action on V. In terms of the embedding $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$, the proof of Proposition 1.8 shows that the action of T_N on $Y_{\mathscr{A}}$ is induced by the usual action of $(\mathbb{C}^*)^s$ on $\mathbb{C}^s$. But how do we see the action instrinsically, without embedding into affine space? This is where semigroup homomorphisms prove their value. Fix $t \in T_N$ and $p \in V$, and let p

correspond to the semigroup homorphism $m \mapsto \gamma(m)$. In Exercise 3.1 you will show that $t \cdot p$ is given by the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$. This abstract description will prove useful in Chapter 3 when we study the orbits of the torus action.

From the point of view of group actions, the action of T_N on V is given by a map $T_N \times V \to V$. Since both sides are affine varieties, this should be a morphism, meaning that it should come from a $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[\mathsf{S}] = \mathbb{C}[V] \longrightarrow \mathbb{C}[T_N \times V] = \mathbb{C}[T_N] \otimes_{\mathbb{C}} \mathbb{C}[V] = \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}].$$

This homomorphism is given by $\chi^m \mapsto \chi^m \otimes \chi^m$ for $m \in \mathsf{S}$ (Exercise 3.2).

We can also characterize those affine toric varieties for which the torus action has a fixed point. We say that an affine semigroup S is *pointed* if $\mathsf{S} \cap (-\mathsf{S}) = \{0\}$, i.e., if 0 is the only element of S with an inverse. This is the semigroup analog of being strongly convex.

Proposition 3.2. *Let V be the affine toric variety. Then:*

(a) *If we write $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$, then the torus action has a fixed point if and only if S is pointed, in which case the unique fixed point is given by the the semigroup homomorphism $\mathsf{S} \to \mathbb{C}$ defined by*

$$m \longmapsto \begin{cases} 1 & m = 0 \\ 0 & m \neq 0. \end{cases} \tag{3.1}$$

(b) *If we write $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ for $\mathscr{A} \subseteq \mathsf{S} \setminus \{0\}$, then the torus action has a fixed point if and only $0 \in Y_{\mathscr{A}}$, in which case the unique fixed point is 0.*

Proof. For part (a), let $p \in V$ be represented by the semigroup homomorphism $\gamma : \mathsf{S} \to \mathbb{C}$. Then p is fixed by the torus action if and only if $\chi^m(t)\gamma(m) = \gamma(m)$ for all $m \in \mathsf{S}$ and $t \in T_N$. This equation is satisfied for $m = 0$ since $\gamma(0) = 1$, and if $m \neq 0$, then picking t with $\chi^m(t) \neq 0$ shows that $\gamma(m) = 0$. Thus, if a fixed point exists, then it is unique and is given by (3.1). Then we are done since (3.1) is a semigroup homomorphism if and only if S is pointed.

For part (b), first assume that $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ has a fixed point, in which case $\mathsf{S} = \mathbb{N}\mathscr{A}$ is pointed and the unique point p is given by (3.1). Then $\mathscr{A} \subseteq \mathsf{S} \setminus \{0\}$ and the proof of Proposition 3.1 imply that p is the origin in $\mathbb{C}^s$, so that $0 \in Y_{\mathscr{A}}$. The converse follows since $0 \in \mathbb{C}^s$ is fixed by $(\mathbb{C}^*)^s$ hence by $T_N \subseteq (\mathbb{C}^*)^s$. □

When $V = U_\sigma$, we can state Proposition 3.2 as follows (Exercise 3.3).

Corollary 3.3. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex polyhedral cone. Then the torus action on U_σ has a fixed point if and only if $\dim \sigma = \dim N_{\mathbb{R}}$, in which case the fixed point is unique and is given by the maximal ideal*

$$\langle \chi^m \mid m \in \mathsf{S}_\sigma \setminus \{0\} \rangle \subseteq \mathbb{C}[\mathsf{S}_\sigma],$$

where as usual $\mathsf{S}_\sigma = \sigma^\vee \cap M$. □

We will see in Chapter 3 that this corollary is part of the correspondence between torus orbits of U_σ and faces of σ.

Normality and Saturation. We next study the question of when an affine toric variety V is normal. We need one definition before stating our normality criterion.

Definition 3.4. An affine semigroup $\mathsf{S} \subseteq M$ is ***saturated*** if for all $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, $km \in \mathsf{S}$ implies $m \in \mathsf{S}$.

For example, if $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone, then $\mathsf{S}_\sigma = \sigma^\vee \cap M$ is easily seen to be saturated (Exercise 3.4).

Theorem 3.5. *Let V be an affine toric variety with torus T_N. Then the following are equivalent:*

(a) V *is normal.*

(b) $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$*, where* $\mathsf{S} \subseteq M$ *is a saturated affine semigroup.*

(c) $V = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ $(= U_\sigma)$*, where* $\mathsf{S}_\sigma = \sigma^\vee \cap M$ *and* $\sigma \subseteq N_\mathbb{R}$ *is a strongly convex rational polyhedral cone.*

Proof. By Theorem 1.16, $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S contained in a lattice, and by Proposition 1.14, the torus of V has the character lattice $M = \mathbb{Z}\mathsf{S}$. Also let $n = \dim V$, so that $M \simeq \mathbb{Z}^n$. We will use this to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b): If V is normal, then $\mathbb{C}[\mathsf{S}] = \mathbb{C}[V]$ is integrally closed in its field of fractions $\mathbb{C}(V)$. Suppose that $km \in \mathsf{S}$ for some $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$. Then χ^m is a polynomial function on T_N and hence a rational function on V since $T_N \subseteq V$ is Zariski open. We also have $\chi^{km} \in \mathbb{C}[\mathsf{S}]$ since $km \in \mathsf{S}$. It follows that χ^m is a root of the monic polynomial $X^k - \chi^{km}$ with coefficients in $\mathbb{C}[\mathsf{S}]$. By the definition of normal, we obtain $\chi^m \in \mathbb{C}[\mathsf{S}]$, i.e., $m \in \mathsf{S}$. Thus S is saturated.

(b) $\Rightarrow$ (c): Let $\mathscr{A} \subseteq \mathsf{S}$ be a finite generating set of S. Then S lies in the rational polyhedral cone $\mathrm{Cone}(\mathscr{A}) \subseteq M_\mathbb{R}$, and $\mathrm{rank}\, \mathbb{Z}\mathscr{A} = n$ implies $\dim \mathrm{Cone}(\mathscr{A}) = n$ by Exercise 2.3. It follows that $\sigma = \mathrm{Cone}(\mathscr{A})^\vee \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone such that $\mathsf{S} \subseteq \sigma^\vee \cap M$. In Exercise 3.4 you will prove that equality holds when S is saturated. Hence $\mathsf{S} = \mathsf{S}_\sigma$.

(c) $\Rightarrow$ (a): We need to show that $\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ is normal when $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone. Let $\rho_1, \ldots, \rho_r$ be the rays of σ. Since σ is generated by its rays (Lemma 2.13), we have

$$\sigma^\vee = \bigcap_{i=1}^r \rho_i^\vee,$$

and intersecting with M gives

$$\mathsf{S}_\sigma = \bigcap_{i=1}^r \mathsf{S}_{\rho_i}.$$

By Exercise 0.5, S_σ is normal if each S_{ρ_i} is normal, so it suffices to prove that S_ρ is normal when ρ is a rational ray in $N_\mathbb{R}$. Let $u \in \rho \cap N$ be the ray generator of ρ. Since u is *primitive*, i.e, $\frac{1}{k}u \notin N$ for all $k > 1$, we can find a basis $e_1, \dots, e_n$ of N with $u = e_1$ (Exercise 3.5). Thus we can assume that $\rho = \mathrm{Cone}(e_1)$, so that

$$\mathbb{C}[\mathsf{S}_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

by Example 2.18. But $\mathbb{C}[x_1, \dots, x_n]$ is normal (it is a UFD), so its localization

$$\mathbb{C}[x_1, \dots, x_n]_{x_2 \cdots x_n} = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

is also normal by Exercise 0.5. This completes the proof. □

Example 3.6. We saw in Example 2.17 that $V = \mathbf{V}(xy - zw)$ is the affine toric variety U_σ of the cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$ pictured in Figure 1. Then Theorem 3.5 implies that V is normal, as claimed in Example 1.5. ◇

Example 3.7. By Example 2.19, the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ is the affine toric variety of $\sigma = \mathrm{Cone}(de_1 - e_2, e_1) \subseteq \mathbb{R}^2$ and hence is normal by Theorem 3.5.

It is instructive to view this example using the parametrization

$$\Phi_{\mathscr{A}}(s,t) = (s^d, s^{d-1}t, \dots, st^{d-1}, t^d)$$

from Example 1.6. Plotting the lattice points in $\mathscr{A}$ for $d = 2$ gives the white squares in Figure 9 (a). These generate the semigroup $\mathsf{S} = \mathbb{N}\mathscr{A}$, and the proof of Theorem 3.5 gives the cone $\sigma^\vee = \mathrm{Cone}(e_1, e_2)$, which is the first quadrant in the figure.

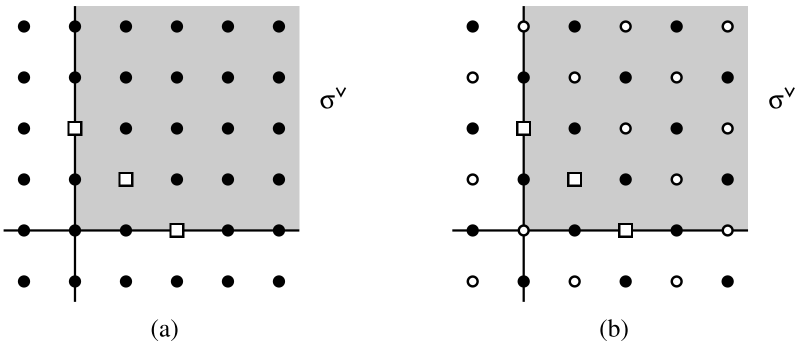


Figure 9. Lattice points for the rational normal cone $\widehat{C}_2$

At first glance, something seems wrong. The affine variety $\widehat{C}_2$ is normal, yet in Figure 9 (a) the semigroup generated by the white squares misses some lattice points in $\sigma^\vee$. This semigroup does not look saturated. How can the affine toric variety be normal?

The problem is that we are using the wrong lattice! Proposition 1.8 tells us to use the lattice $\mathbb{Z}\mathscr{A}$, which gives the white dots and squares in Figure 9 (b). This figure shows that the white squares generate the semigroup of lattice points in $\sigma^\vee$. Hence S is saturated and everything is fine. ◇

This example points out the importance of working with the correct lattice.

The Normalization of an Affine Toric Variety. The normalization of an affine toric variety is easy to describe. Let $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S, so that the torus of the V has character lattice $M = \mathbb{Z}\mathsf{S}$. Let $\mathrm{Cone}(\mathsf{S})$ denote the cone of any finite generating set of S and set $\sigma = \mathrm{Cone}(\mathsf{S})^\vee \subseteq N_\mathbb{R}$. In Exercise 3.6 you will prove the following.

Proposition 3.8. *The above cone σ is a strongly convex rational polyhedral cone in $N_\mathbb{R}$ and the inclusion $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[\sigma^\vee \cap M]$ induces a morphism $U_\sigma \to V$ that is the normalization map of V.* □

The normalization of an affine toric variety of the form $Y_\mathscr{A}$ is constructed by applying Proposition 3.8 to the affine semigroup $\mathbb{N}\mathscr{A}$ and the lattice $\mathbb{Z}\mathscr{A}$.

Example 3.9. Let $\mathscr{A} = \{(4,0),(3,1),(1,3),(0,4)\} \subseteq \mathbb{Z}^2$. Then

$$\Phi_\mathscr{A}(s,t) = (s^4, s^3t, st^3, t^4)$$

parametrizes the surface $Y_\mathscr{A} \subseteq \mathbb{C}^4$ considered in Exercise 1.7. This is almost the rational normal cone $\widehat{C}_4$, except that we have omitted s^2t^2. Using $(2,2) = \frac{1}{2}\big((4,0)+(0,4)\big)$, we see that $\mathbb{N}\mathscr{A}$ is not saturated, so that $Y_\mathscr{A}$ is not normal.

Applying Proposition 3.8, one sees that the normalization of $Y_\mathscr{A}$ is $\widehat{C}_4$. This is an affine variety in $\mathbb{C}^5$, and the normalization map is induced by the obvious projection $\mathbb{C}^5 \to \mathbb{C}^4$. ◇

In Chapter 3 we will show that the normalization map $U_\sigma \to V$ constructed in Proposition 3.8 is one-to-one and onto.

Smooth Affine Toric Varieties. Our next goal is to characterize when an affine toric variety is smooth. Since smooth affine varieties are normal (Proposition 0.9), we need only consider toric varieties U_σ coming from strongly convex rational polyhedral cones $\sigma \subseteq N_\mathbb{R}$.

We begin by studying U_σ when σ has maximal dimension. Then $\sigma^\vee$ is strongly convex, so that $\mathsf{S}_\sigma = \sigma^\vee \cap M$ has a Hilbert basis $\mathscr{H}$. Furthermore, Corollary 3.3 tells us that torus action on U_σ has a unique fixed point, denoted here by $p_\sigma \in U_\sigma$. The point p_σ and the Hilbert basis $\mathscr{H}$ are related as follows.

Lemma 3.10. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone of maximal dimension and let $T_{p_\sigma}(U_\sigma)$ be the Zariski tangent space to the affine toric variety U_σ at the above point p_σ. Then $\dim T_{p_\sigma}(U_\sigma) = |\mathscr{H}|$.*

Proof. By Corollary 3.3, the maximal ideal of $\mathbb{C}[S_\sigma]$ corresponding to p_σ is $\mathfrak{m} = \langle \chi^m \mid m \in \mathsf{S}_\sigma \setminus \{0\}\rangle$. Since $\{\chi^m\}_{m\in\mathsf{S}_\sigma}$ is basis of $\mathbb{C}[S_\sigma]$, we obtain

$$\mathfrak{m} = \bigoplus_{m\neq 0} \mathbb{C}\chi^m = \bigoplus_{m \text{ irreducible}} \mathbb{C}\chi^m \;\oplus \bigoplus_{m \text{ reducible}} \mathbb{C}\chi^m = \Big(\bigoplus_{m\in\mathscr{H}} \mathbb{C}\chi^m\Big) \oplus \mathfrak{m}^2.$$

It follows that $\dim \mathfrak{m}/\mathfrak{m}^2 = |\mathscr{H}|$. To relate this to the maximal ideal $\mathfrak{m}_{U_\sigma,p_\sigma}$ in the local ring $\mathscr{O}_{U_\sigma,p_\sigma}$, we use the natural map

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}_{U_\sigma,p_\sigma}/\mathfrak{m}^2_{U_\sigma,p_\sigma}$$

which is always an isomorphism (Exercise 3.7). Since $T_{p_\sigma}(U_\sigma)$ is the dual space of $\mathfrak{m}_{U_\sigma,p_\sigma}/\mathfrak{m}^2_{U_\sigma,p_\sigma}$, we see that $\dim T_{p_\sigma}(U_\sigma) = |\mathscr{H}|$. □

The Hilbert basis $\mathscr{H}$ of S_σ gives $U_\sigma = Y_{\mathscr{H}} \subseteq \mathbb{C}^s$, where $s = |\mathscr{H}|$. This affine embedding is especially nice. Given *any* affine embedding $U_\sigma \hookrightarrow \mathbb{C}^\ell$, we have $\dim T_{p_\sigma}(U_\sigma) \le \ell$ by Lemma 0.6. In other words, $\dim T_{p_\sigma}(U_\sigma)$ is a lower bound on the dimension of an affine embedding. Then Lemma 3.10 shows that when σ has maximal dimension, the Hilbert basis of S_σ gives the most efficient affine embedding of U_σ.

Example 3.11. In Example 2.19, we saw that the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ is the toric variety coming from $\sigma = \mathrm{Cone}(e_1, de_1 - e_2) \subseteq \mathbb{R}^2$ and that $\mathsf{S}_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ is generated by $(1,i)$ for $0 \le i \le d$. These generators form the Hilbert basis of S_σ, so that the Zariski tangent space of $0 \in \widehat{C}_d$ has dimension $d+1$. Hence $\mathbb{C}^{d+1}$ in the smallest affine space in which we can embed $\widehat{C}_d$. ◇

We now come to our main result about smoothness. Recall from §2 that a rational polyhedral cone is *smooth* if it can be generated by a subset of a basis of the lattice.

Theorem 3.12. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex polyhedral cone. Then U_σ is smooth if and only if σ is smooth. Furthermore, all smooth affine toric varieties are of this form.*

Proof. If an affine toric variety is smooth, then it is normal by Proposition 0.9 and hence of the form U_σ. Also, Example 2.18 implies that if σ is smooth as a cone, then U_σ is smooth as a variety. It remains to prove the converse. So fix $\sigma \subseteq N_\mathbb{R}$ such that U_σ is smooth. Let $n = \dim U_\sigma = \dim N_\mathbb{R}$.

First suppose that σ has dimension n and let $p_\sigma \in U_\sigma$ be the point studied in Lemma 3.10. Since p_σ is smooth in U_σ, the Zariski tangent space $T_{p_\sigma}(U_\sigma)$ has dimension n by Definition 0.7. On the other hand, Lemma 3.10 implies that $\dim T_{p_\sigma}(U_\sigma)$ is the cardinality of the Hilbert basis $\mathscr{H}$ of $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Thus

$$n = |\mathscr{H}| \ge |\{\text{edges } \rho \subseteq \sigma^\vee\}| \ge n,$$

where the first inequality holds by Proposition 2.20 (each edge $\rho \subseteq \sigma^\vee$ contributes an element of $\mathscr{H}$) and the second holds since $\dim \sigma^\vee = n$. It follows that σ

has n edges and $\mathscr{H}$ consists of the ray generators of these edges. Since $M = \mathbb{Z}\mathsf{S}_\sigma$ by (2.3), the n edge generators of $\sigma^\vee$ generate the lattice $M \simeq \mathbb{Z}^n$ and hence form a basis of M. Thus $\sigma^\vee$ is smooth, and then $\sigma = (\sigma^\vee)^\vee$ is smooth since duality preserves smoothness.

Next suppose that $\dim\sigma = r < n$. We reduce to the previous case as follows. Let $N_1 \subseteq N$ be the smallest saturated sublattice containing the generators of σ. Then N/N_1 is torsion-free, which by Exercise 3.5 implies the existence of a sublattice $N_2 \subseteq N$ with $N = N_1 \oplus N_2$. Note that $\operatorname{rank} N_1 = r$ and $\operatorname{rank} N_2 = n - r$.

The cone σ lies in both $(N_1)_\mathbb{R}$ and $N_\mathbb{R}$. This gives affine toric varieties U_{σ,N_1} and $U_{\sigma,N}$ of dimensions r and n respectively. Furthermore, $N = N_1 \oplus N_2$ induces $M = M_1 \oplus M_2$, so that $\sigma \subseteq (N_1)_\mathbb{R}$ and $\sigma \subseteq N_\mathbb{R}$ give the affine semigroups $\mathsf{S}_{\sigma,N_1} \subseteq M_1$ and $\mathsf{S}_{\sigma,N} \subseteq M$ respectively. It is straighforward to show that

$$\mathsf{S}_{\sigma,N} = \mathsf{S}_{\sigma,N_1} \oplus M_2,$$

which in terms of semigroup algebras can be written

$$\mathbb{C}[\mathsf{S}_{\sigma,N}] \simeq \mathbb{C}[\mathsf{S}_{\sigma,N_1}] \otimes_\mathbb{C} \mathbb{C}[M_2].$$

The right-hand side is the coordinate ring of $U_{\sigma,N_1} \times T_{N_2}$. Thus

$$U_{\sigma,N} \simeq U_{\sigma,N_1} \times T_{N_2}, \tag{3.2}$$

which in turn implies that

$$U_{\sigma,N} \simeq U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r} \subseteq U_{\sigma,N_1} \times \mathbb{C}^{n-r}.$$

Since we are assuming that $U_{\sigma,N}$ is smooth, it follows that $U_{\sigma,N_1} \times \mathbb{C}^{n-r}$ is smooth at any point (p,q) in $U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r}$. In Exercise 3.8 you will show that

$$U_{\sigma,N_1} \times \mathbb{C}^{n-r} \text{ is smooth at } (p,q) \implies U_{\sigma,N_1} \text{ is smooth at } p. \tag{3.3}$$

Letting $p = p_\sigma \in U_{\sigma,N_1}$, the previous case implies that σ is smooth in N_1 since $\dim\sigma = \dim(N_1)_\mathbb{R}$. Hence σ is clearly smooth in $N = N_1 \oplus N_2$. □

Equivariant Maps between Affine Toric Varieties. We next study maps $V_1 \to V_2$ between affine toric varieties that respect the torus actions on V_1 and V_2.

Definition 3.13. Let $V_i = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$ be the affine toric varieties coming from the affine semigroups S_i, $i = 1, 2$. Then a morphism $\varphi : V_1 \to V_2$ is ***toric*** if the corresponding map of coordinate rings $\varphi^* : \mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$ is induced by a semigroup homomorphism $\widehat{\varphi} : \mathsf{S}_2 \to \mathsf{S}_1$.

Here is our first result concerning toric morphisms.

Proposition 3.14. *Let T_{N_i} be the torus of the affine toric variety V_i, $i = 1, 2$. Then:*

(a) *A morphism $\varphi : V_1 \to V_2$ is toric if and only if*

$$\varphi(T_{N_1}) \subseteq T_{N_2}$$

and $\varphi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.

(b) *A toric morphism $\varphi : V_1 \to V_2$ is **equivariant**, meaning that*

$$\varphi(t \cdot p) = \varphi(t) \cdot \varphi(p)$$

for all $t \in T_{N_1}$ and $p \in V_1$.

Proof. Let $V_i = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$, so that the character lattice of T_{N_i} is $M_i = \mathbb{Z}\mathsf{S}_i$. If φ comes from a semigroup homomorphism $\widehat{\varphi} : \mathsf{S}_2 \to \mathsf{S}_1$, then $\widehat{\varphi}$ extends to a group homomorphism $\widehat{\varphi} : M_2 \to M_1$ and hence gives a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\mathsf{S}_2] & \xrightarrow{\varphi^*} & \mathbb{C}[\mathsf{S}_1] \\ \downarrow & & \downarrow \\ \mathbb{C}[M_2] & \longrightarrow & \mathbb{C}[M_1]. \end{array}$$

Applying Spec, we see that $\varphi(T_{N_1}) \subseteq T_{N_2}$, and $\varphi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism since $T_{N_i} = \operatorname{Hom}_{\mathbb{Z}}(M_i, \mathbb{C}^*)$ by Exercise 1.10. Conversely, if φ satisfies these conditions, then $\varphi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ induces a diagram as above where the bottom map comes from a group homomorphism $\widehat{\varphi} : M_2 \to M_1$. This, combined with $\varphi^*(\mathbb{C}[\mathsf{S}_2]) \subseteq \mathsf{S}_1$, implies that $\widehat{\varphi}$ induces a semigroup homomorphism $\widehat{\varphi} : \mathsf{S}_2 \to \mathsf{S}_1$. This proves part (a) of the proposition.

For part (b), suppose that we have a toric map $\varphi : V_1 \to V_2$. The action of T_{N_i} on V_i is given by a morphism $\Phi_i : T_{N_i} \times V_i \longrightarrow V_i$, and equivariance means that we have a commutative diagram

$$\begin{array}{ccc} T_{N_1} \times V_1 & \xrightarrow{\Phi_1} & V_1 \\ {\scriptstyle \varphi|_{T_{N_1}} \times \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\ T_{N_2} \times V_2 & \xrightarrow{\Phi_2} & V_2. \end{array}$$

If we replace V_i with T_{N_i} in the diagram, then it certainly commutes since $\varphi|_{T_{N_1}}$ is a group homomorphism. Then the whole diagram commutes since $T_{N_1} \times T_{N_1}$ is Zariski dense in $T_{N_1} \times V_1$. □

We can also characterize toric morphisms between affine toric varieties coming from strongly convex rational polyhedral cones. First note that a homomorphism $\bar{\varphi} : N_1 \to N_2$ of lattices gives a group homomorphism $\varphi : T_{N_1} \to T_{N_2}$ of tori. This follows from $T_{N_i} = N_i \otimes_{\mathbb{Z}} \mathbb{C}^*$, and one sees that φ is a morphism. Also, tensoring $\bar{\varphi}$ with $\mathbb{R}$ gives $\bar{\varphi}_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$

Here is the result, whose proof we leave to the reader (Exercise 3.9).

Proposition 3.15. *Suppose that we have strongly convex rational polyhedral cones $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ and a homomorphism $\bar{\varphi} : N_1 \to N_2$. Then $\varphi : T_{N_1} \to T_{N_2}$ extends to a map of affine toric varieties $\varphi : U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\bar{\varphi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.* □

In the remainder of the chapter we explore some interesting classes of toric morphisms.

Faces and Affine Open Subsets. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and let $\tau \subseteq \sigma$ be a face. Then we can find $m \in \sigma^\vee \cap M$ such that $\tau = H_m \cap \sigma$. This allows us to relate semigroup algebras of σ and τ as follows.

Proposition 3.16. *Let $\tau = H_m \cap \sigma$, $m \in \sigma^\vee \cap M$, be a face of σ as above. Then the semigroup algebra $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\tau^\vee \cap M]$ is the localization of $\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ at $\chi^m \in \mathbb{C}[\mathsf{S}_\sigma]$. In other words,*

$$\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}.$$

Proof. The inclusion $\tau \subseteq \sigma$ implies $\mathsf{S}_\sigma \subset \mathsf{S}_\tau$, and since $\langle m, u\rangle = 0$ for all $u \in \tau$, we have $\pm m \in \tau^\vee$. It follows that

$$\mathsf{S}_\sigma + \mathbb{Z}(-m) \subseteq \mathsf{S}_\tau.$$

This inclusion is actually an equality, as we now prove. Fix a finite set $S \subset M$ with $\sigma = \mathrm{Cone}(S)$ and pick $m' \in \mathsf{S}_\tau$. Set

$$C = \max_{u \in S}\{|\langle m', u\rangle|\} \in \mathbb{N}.$$

It is straightforward to show that $m' + Cm \in \mathsf{S}_\sigma$. This proves that

$$\mathsf{S}_\sigma + \mathbb{Z}(-m) = \mathsf{S}_\tau,$$

from which $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}$ follows immediately. $\square$

We can interpret this in terms of toric morphisms as follows. by Proposition 3.15, the identity map $N \to N$ and the inclusion $\tau \subseteq \sigma$ give the toric morphism $U_\tau \to U_\sigma$ that corresponds to the inclusion $\mathbb{C}[\mathsf{S}_\sigma]) \subseteq \mathbb{C}[\mathsf{S}_\tau]$. By Proposition 3.16,

(3.4) $U_\tau = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\tau]) = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}) = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma])_{\chi^m} = (U_\sigma)_{\chi^m} \subseteq U_\sigma.$

In other words, U_τ becomes an affine open subset of U_σ when τ is a face of σ. This will be useful in Chapters 2 and 3 when we study the local structure of more general toric varieties.

Sublattices of Finite Index and Rings of Invariants. Another interesting class of toric morphisms arises when we keep the same cone but change the lattice. Here is an example we have already seen.

Example 3.17. In Example 3.7 the dual of $\sigma = \mathrm{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$ interacts with the lattices shown in Figure 10 on the next page. To make this precise, let us name the lattices involved: the lattices

$$N' = \mathbb{Z}^2 \subseteq N = \{(a/2, b/2) \mid a, b \in \mathbb{Z},\ a + b \equiv 0 \bmod 2\}$$

have $\sigma \subseteq N'_{\mathbb{R}} \subseteq N_{\mathbb{R}}$, and the dual lattices

$$M' = \mathbb{Z}^2 \supseteq M = \{(a, b) \mid a, b \in \mathbb{Z},\ a + b \equiv 0 \bmod 2\}$$

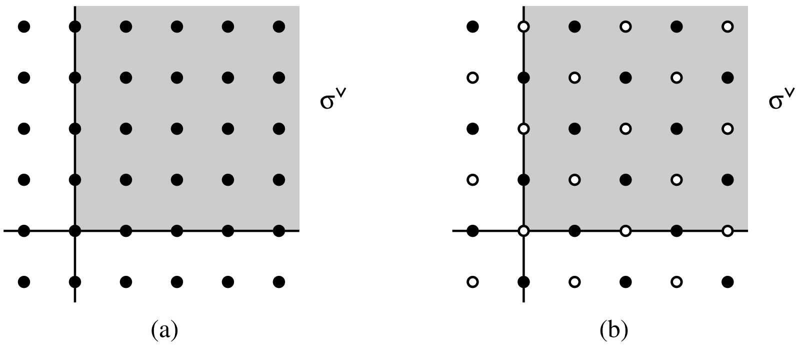


Figure 10. Lattice points of $\sigma^\vee$ relative to two lattices

have $\sigma^\vee \subseteq M'_{\mathbb{R}} \subseteq M_{\mathbb{R}}$. Note that duality reverses inclusions and that M and N are indeed dual under dot product. In Figure 10 (a), the black dots in the first quadrant form the semigroup $\mathsf{S}_{\sigma,N'} = \sigma^\vee \cap M'$, and in Figure 10 (b), the white dots in the first quadrant form $\mathsf{S}_{\sigma,N} = \sigma^\vee \cap M$.

This gives the affine toric varieties $U_{\sigma,N'}$ and $U_{\sigma,N}$. Clearly $U_{\sigma,N'} = \mathbb{C}^2$ since σ is smooth for N', while Example 3.7 shows that $U_{\sigma,N}$ is the rational normal cone $\widehat{C}_2$. The inclusion $N' \subset N$ gives a toric morphism

$$\mathbb{C}^2 = U_{\sigma,N'} \longrightarrow U_{\sigma,N} = \widehat{C}_2.$$

Our next task is to find a nice description of this map. ◇

In general, suppose we have lattices $N' \subseteq N$, where N' has finite index in N, and let $\sigma \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Then the inclusion $N' \subseteq N$ gives the toric morphism

$$\varphi : U_{\sigma,N'} \longrightarrow U_{\sigma,N}.$$

The dual lattices satisfy $M' \supseteq M$, so that φ corresponds to the inclusion

$$\mathbb{C}[\mathsf{S}_{\sigma,N'}] = \mathbb{C}[\sigma^\vee \cap M'] \supseteq \mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\mathsf{S}_{\sigma,N}]$$

of semigroup algebras. The idea is to realize $\mathbb{C}[\mathsf{S}_{\sigma,N}]$ as a ring of invariants of a group action on $\mathbb{C}[\mathsf{S}_{\sigma,N'}]$. The group in question is the finite abelian group

$$G = \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*), \tag{3.5}$$

which acts on $\mathbb{C}[\mathsf{S}_{\sigma,N'}]$ via

$$g \cdot \chi^{m'} = g([m'])\, \chi^{m'}.$$

One can prove that $m' \in M'$ lies in M if and only if $g([m']) = 1$ for all $g \in G$. Hence the ring of invariants

$$\mathbb{C}[\mathsf{S}_{\sigma,N'}]^G = \{f \in \mathbb{C}[\mathsf{S}_{\sigma,N'}] \mid g \cdot f = f \text{ for all } g \in G\},$$

is precisely $\mathbb{C}[\mathsf{S}_{\sigma,N}]$, i.e.,

$$\mathbb{C}[\mathsf{S}_{\sigma,N'}]^G = \mathbb{C}[\mathsf{S}_{\sigma,N}].$$

In terms of the corresponding affine toric varieties, this has the following geometric consequence.

Proposition 3.18. *If N' has finite index in N and $\sigma \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone, then the finite group G defined in* (3.5) *acts on $U_{\sigma,N'}$ and the toric morphism $U_{\sigma,N'} \to U_{\sigma,N}$ induces a bijection*

$$U_{\sigma,N'}/G \simeq U_{\sigma,N}.$$

Proof. When a finite group G acts on an affine variety $V = \operatorname{Spec}(R)$ over $\mathbb{C}$, we get a set V/G of G-orbits as well as an action of G on the coordinate ring R. The key fact is that V/G has a natural structure as an affine variety, namely

$$V/G \simeq \operatorname{Spec}(R^G),$$

where R^G is the ring of invariants of the G-action on R. When $V = \mathbb{C}^n$ and G acts linearly, this is proved in [**6**, Theorem 10 of Chapter 7]. This proof adapts easily to the case when $R = \mathbb{C}[\mathsf{S}_{\sigma,N'}]$ (Exercise 3.10). □

Here is an example of Proposition 3.18.

Example 3.19. In the situation of Example 3.17, one computes that G is the group $\mu_2 = \{\pm 1\}$ acting on $U_{\sigma,N'} = \operatorname{Spec}(\mathbb{C}[s,t]) \simeq \mathbb{C}^2$ by $-1 \cdot (s,t) = (-s,-t)$. Thus the rational normal cone $\widehat{C}_2$ is the quotient

$$\mathbb{C}^2/\mu_2 = U_{\sigma,N'}/\mu_2 \simeq U_{\sigma,N} = \widehat{C}_2.$$

We can see this explicitly as follows. The invariant ring is easily seen to be

$$\mathbb{C}[s,t]^{\mu_2} = \mathbb{C}[s^2, st, t^2] = \mathbb{C}[\widehat{C}_2] \simeq \mathbb{C}[x_0,x_1,x_2]/\langle x_0x_2 - x_1^2\rangle,$$

where the last isomorphism follows from Example 1.6. From the point of view of invariant theory, the generators s^2, st, t^2 of the ring of invariants give a morphism

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^3, \quad (s,t) \to (s^2, st, t^2)$$

that is constant on μ_2-orbits. This map also separates orbits, so it induces

$$\mathbb{C}^2/\mu_2 \simeq \Phi(\mathbb{C}^2) = \widehat{C}_2,$$

where the last equality is by Example 1.6. But we can also think about this in terms of semigroups, where the exponent vectors of s^2, st, t^2 give the Hilbert basis of the semigroup $\mathsf{S}_{\sigma,N}$ pictured in Figure 11 (b). Everything fits together very nicely. ◇

In Exercise 3.11 you will generalize Example 3.19 to the case of the rational normal cone $\widehat{C}_d$ for arbitrary d.

The group G from (3.5) has some interesting properties (Exercise 3.12).

Proposition 3.20. *Let N be a lattice and $N' \subseteq N$ a sublattice of finite index, with dual lattices $M' \supseteq M$. Define $G = \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*)$ as in* (3.5). *Then:*

(a) *There is a perfect bilinear pairing $N/N' \times M'/M \to \mathbb{C}^*$ that gives a canonical isomorphism $N/N' \simeq G$.*

(b) *G is the kernel of the map of tori $T_{N'} \to T_N$ induced by $N' \subseteq N$.*

Here is our final example of Proposition 3.18.

Example 3.21. Let $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a simplicial cone of dimension n with ray generators $u_1, \ldots, u_n$. Then $N' = \sum_{i=1}^n \mathbb{Z}u_i$ is a sublattice of finite index. Furthermore, σ is smooth relative to N', so that $U_{\sigma,N'} = \mathbb{C}^n$. It follows that $G = N/N'$ acts on $\mathbb{C}^n$ with quotient

$$\mathbb{C}^n/G = U_{\sigma,N'}/G \simeq U_{\sigma,N}.$$

Hence the affine toric variety of a simplicial cone is the quotient of affine space by a finite abelian group. In the literature, varieties like $U_{\sigma,N}$ are called *orbifolds* and are said to be $\mathbb{Q}$*-factorial*.

Exercises for §3.

3.1. Consider the affine toric variety $Y_{\mathscr{A}} = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$, where $\mathscr{A} = \{m_1, \ldots, m_s\}$ and $\mathsf{S} = \mathbb{N}\mathscr{A}$. Let $\gamma : \mathsf{S} \to \mathbb{C}$ be a semigroup homomorphism. In the proof of Proposition 3.1 we showed that $p = (\gamma(m_1), \ldots, \gamma(m_s))$ lies in $Y_{\mathscr{A}}$.

(a) Prove that the maximal ideal $\{f \in \mathbb{C}[\mathsf{S}] \mid f(p) = 0\}$ is the kernel of the $\mathbb{C}$-algebra homomophism $\mathbb{C}[\mathsf{S}] \to \mathbb{C}$ induced by γ.

(b) The torus T_N of $Y_{\mathscr{A}}$ has character lattice $M = \mathbb{Z}\mathscr{A}$ and fix $t \in T_N$. As in the discussion following Proposition 3.1, this gives the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$. Prove that this corresponds to the point

$$(\chi^{m_1}, \ldots, \chi^{m_s}) \cdot (\gamma(m_1), \ldots, \gamma(m_s)) = (\chi^{m_1}\gamma(m_1), \ldots, \chi^{m_s}\gamma(m_s))$$

coming from the action of $t \in T_N \subseteq (\mathbb{C}^*)^s$ on $p \in Y_{\mathscr{A}} \subseteq \mathbb{C}^s$.

3.2. Let $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ with for torus $T_N = \mathrm{Spec}(\mathbb{C}[M])$, $M = \mathbb{Z}\mathsf{S}$. The action $T_N \times V \to V$ comes from a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\mathsf{S}] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}]$. Prove that this homomorphism is given by $\chi^m \mapsto \chi^m \otimes \chi^m$. Hint: First show that this formula determines the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]$ that gives the group operation $T_N \times T_N \to T_N$.

3.3. Prove Corollary 3.3.

3.4. Let $\mathscr{A} \subseteq M$ be a finite set.

(a) Prove that the semigroup $\mathbb{N}\mathscr{A}$ is saturated in M if and only if $\mathbb{N}\mathscr{A} = \mathrm{Cone}(\mathscr{A}) \cap M$. Hint: Apply (2.2) to $\mathrm{Cone}(\mathscr{A}) \subseteq M_{\mathbb{R}}$.

(b) Complete the proof of (b) ⇒ (c) from Theorem 3.5.

3.5. Let N be a lattice.

(a) Let $N_1 \subseteq N$ be a sublattice such that N/N_1 is torsion-free. Prove that there is a sublattice $N_2 \subseteq N$ such that $N = N_1 \oplus N_2$.

(b) Let $u \in N$ be primitive as defined in the proof of Theorem 3.5. Prove that N has a basis $e_1, \ldots, e_n$ such that $e_1 = u$.

3.6. Prove Proposition 3.8.

3.7. Let p be a point of an irreducible affine variety V. Then p gives the maximal ideal $\mathfrak{m} = \{f \in \mathbb{C}[V] \mid f(p) = 0\}$ as well as the maximal ideal $\mathfrak{m}_{V,p} \subseteq \mathscr{O}_{V,p}$ defined in §0. Prove that the natural map $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2$ is an isomorphism of $\mathbb{C}$-vector spaces.

3.8. Prove (3.3). Hint: Use Lemma 0.6 and Example 0.10.

3.9. Prove Proposition 3.15.

3.10. Prove Proposition 3.18.

3.11. The group of dth roots of unity $\mu_d = \{\zeta \in \mathbb{C}^* \mid \zeta^d = 1\}$ be acts on $\mathbb{C}^2$ by $\zeta \cdot (x, y) = (\zeta x, \zeta y)$. Adapt Example 3.19 to show that $\mathbb{C}^2/\mu_d \simeq \widehat{C}_d$. Hint: Use lattices $N' = \mathbb{Z}^2 \subseteq N = \{(a/d, b/d) \mid a, b \in \mathbb{Z}, a + b \equiv 0 \bmod d\}$.

3.12. Prove Proposition 3.20. Hint: Use the pairing $\langle\ ,\ \rangle : M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$ and note that $M \subseteq M' \subseteq M_{\mathbb{Q}}$ and $N' \subseteq N \subseteq N_{\mathbb{Q}}$. The map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^*$ given by $[q] \mapsto e^{2\pi i q}$ will also be helpful.

3.13. Let $\sigma_1 \subset (N_1)_{\mathbb{R}}$ and $\sigma_2 \subset (N_2)_{\mathbb{R}}$ be strongly convex rational polyhedral cones. This gives the cone $\sigma_1 \times \sigma_2 \subset (N_1 \oplus N_2)_{\mathbb{R}}$. Prove that $U_{\sigma_1 \times \sigma_2} \simeq U_{\sigma_1} \times U_{\sigma_2}$. Also explain how this result applies to (3.2).

3.14. By Proposition 3.1, a point p of an affine toric variety $V = \operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ is represented by a semigroup homomorphism $\gamma : \mathsf{S} \to \mathbb{C}$. Prove that p lies in the torus of V if and only if γ never vanishes, i.e., $\gamma(m) \neq 0$ for all $m \in \mathsf{S}$.

Chapter 2

Projective Toric Varieties

0. Background: Projective Varieties

Our discussion assumes that the reader is familiar with the elementary theory of projective varieties, at the level of [**6**, Chapter 8].

In Chapter 1, we introduced affine toric varieties. In general, a toric variety is an irreducible variety X over $\mathbb{C}$ containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open set such that the action of $(\mathbb{C}^*)^n$ on itself extends to an action on X. We will learn in Chapter 3 that the concept of "variety" is somewhat subtle. Hence we will defer the formal definition of toric variety until then and instead concentrate on toric varieties that live in projective space $\mathbb{P}^n$, defined by

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*, \tag{0.1}$$

where $\mathbb{C}^*$ acts via homotheties, i.e., $\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n)$ for $\lambda \in \mathbb{C}^*$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$. Thus $(a_0, \ldots, a_n)$ are *homogeneous coordinates* of a point in $\mathbb{P}^n$ and are well-defined up to homothety.

The goal of this chapter is to use from lattice points, polytopes, and polyhedra to create toric varieties that lie in $\mathbb{P}^n$.

Homogeneous Coordinate Rings. A projective variety $V \subseteq \mathbb{P}^n$ is defined by the vanishing of finitely many homogeneous polynomials in the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. The *homogeneous coordinate ring* of V is the quotient ring

$$\mathbb{C}[V] = S/\mathbf{I}(V),$$

where $\mathbf{I}(V)$ is generated by all homogeneous polynomials that vanish on V.

The polynomial ring S is graded by setting $\deg(x_i) = 1$. This gives the decomposition $S = \bigoplus_{d=0}^{\infty} S_d$, where S_d is the vector space of homogeneous polynomials of degree d. Homogeneous ideals decompose similarly, and the above coordinate ring $\mathbb{C}[V]$ inherits a grading where

$$\mathbb{C}[V]_d = S_d/\mathbf{I}(V)_d.$$

The ideal $\mathbf{V}(I) \subseteq S = \mathbb{C}[x_0, \ldots, x_n]$ also defines an affine variety $\widehat{V} \subseteq \mathbb{C}^{n+1}$, called the *affine cone* of V. The variety $\widehat{V}$ satisfies

$$V = (\widehat{V} \setminus \{0\})/\mathbb{C}^*, \tag{0.2}$$

and its coordinate ring is the homogeneous coordinate ring of V, i.e.,

$$\mathbb{C}[\widehat{V}] = \mathbb{C}[V].$$

Example 0.1. In Example 1.6 of Chapter 1 we encountered ideal ideal

$$I = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1 \rangle \subseteq \mathbb{C}[x_0, \ldots, x_d]$$

generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix}.$$

Since I is homogeneous, it defines a projective variety $C_d \subseteq \mathbb{P}^d$ that is the image of the map

$$\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$

defined in homogeneous coordinates by $(s,t) \mapsto (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)$ (see Exercise 1.1 of Chapter 1). This shows that C_d is a curve, called the *rational normal curve* of degree d. Furthermore, the affine cone of C_d is the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ discussed in Example 1.6 of Chapter 1.

We know from Chapter 1 that $\widehat{C}_d$ is an affine toric surface; we will soon see that C_d is a projective toric curve. $\diamond$

Example 0.2. The affine toric variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ studied in Chapter 1 is the affine cone of the rojective surface $V = \mathbf{V}(xy - zw) \subseteq \mathbb{P}^3$. Recall that this surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

given by $(s,t;u,v) \mapsto (su, tv, sv, tu)$. We will see below that $V \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is the projective toric variety coming from the unit square in the plane. $\diamond$

As in the affine case, a projective variety $V \subseteq \mathbb{P}^n$ has the *classical topology*, induced from the usual topology on $\mathbb{P}^n$, and the *Zariski topology*, where the Zariski closed sets are subvarieties of V (meaning projective varieties of $\mathbb{P}^n$ contained in V) and the Zariski open sets are their complements.

Rational Functions on Irreducible Projective Varieties. A homogeneous polynomial $f \in S$ of degree d does not give a function on $\mathbb{P}^n$ since

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n).$$

However, the quotient of two such polynomials $f, g \in S_d$ gives the well-defined function

$$\frac{f}{g} : \mathbb{P}^n \setminus \mathbf{V}(g) \to \mathbb{C}.$$

provided $g \neq 0$. We write this as $f/g : \mathbb{P}^n \dashrightarrow \mathbb{C}$ and say that f/g is a *rational function* on $\mathbb{P}^n$.

More generally, suppose that $V \subseteq \mathbb{P}^n$ is irreducible, and let $f, g \in \mathbb{C}[V] = \mathbb{C}[\widehat{V}]$ be homogeneous of the same degree with $g \neq 0$. Then f and g give functions on the affine cone $\widehat{V}$ and hence an element $f/g \in \mathbb{C}(\widehat{V})$. By (0.2), this induces a rational function $f/g : V \dashrightarrow \mathbb{C}$. Thus

$$\mathbb{C}(V) = \{f/g \in \mathbb{C}(\widehat{V}) \mid f, g \in \mathbb{C}[V] \text{ homogeneous of the same degree, } g \neq 0\}$$

is the field of rational functions on V. It is customary to write the set on the left as $\mathbb{C}(\widehat{V})_0$ since it consists of the degree 0 elements of $\mathbb{C}(\widehat{V})$.

Affine Pieces of Projective Varieties. A projective variety $V \subseteq \mathbb{P}^n$ is a union of Zariski open sets that are affine. To see why, let $U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i)$. Then $U_i \simeq \mathbb{C}^n$ via the map

$$(0.3) \qquad (a_0, \ldots, a_n) \longmapsto (a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i)$$

so that in the notation of Chapter 1, we have

$$U_i = \operatorname{Spec}(\mathbb{C}[x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i]).$$

Then $V \cap U_i$ is a Zariski open subset of V that maps via (0.3) to the affine variety in $\mathbb{C}^n$ defined by the equations

$$(0.4) \qquad f(x_0/x_i, \ldots, x_{i-1}/x_i, 1, x_{i+1}/x_i, \ldots, x_n/x_i) = 0$$

as f varies over all homogeneous polynomials in $\mathbf{I}(V)$.

We call $V \cap U_i$ an *affine piece* of V. These affine pieces cover V since the U_i cover $\mathbb{P}^n$. When V is irreducible, we can describe the coordinate rings of the affine pieces as follows. The variable x_i induces an element $\bar{x}_i \in \mathbb{C}[V] = \mathbb{C}[\widehat{V}] \subseteq \mathbb{C}(\widehat{V})$, so that we get the localization

$$(0.5) \qquad \mathbb{C}[V]_{\bar{x}_i} = \{f/\bar{x}_i^k \mid f \in \mathbb{C}[V],\ k \geq 0\} \subseteq \mathbb{C}(\widehat{V})$$

as in equation (0.2) of Chapter 1. Then

$$(0.6) \qquad (\mathbb{C}[V]_{\bar{x}_i})_0 = \{f/\bar{x}_i^k \in \mathbb{C}[V]_{\bar{x}_i} \mid f \text{ is homogeneous of degree } k\}$$

is the subring of $\mathbb{C}[V]_{\bar{x}_i}$ consisting of all element of degree 0. Note that (0.6) uses only the homogeneous coordinate ring $\mathbb{C}[V]$ and the element $\bar{x}_i$. Yet it gives an affine piece of V as follows.

Lemma 0.3. *Assume that V is irreducible and define U_i and $(\mathbb{C}[V]_{\bar{x}_i})_0$ as above. Then the affine piece $V \cap U_i$ of V has coordinate ring*

$$\mathbb{C}[V \cap U_i] \simeq (\mathbb{C}[V]_{\bar{x}_i})_0.$$

Proof. We have an exact sequence

$$0 \longrightarrow \mathbf{I}(V) \longrightarrow \mathbb{C}[x_0,\ldots,x_n] \longrightarrow \mathbb{C}[V] \longrightarrow 0.$$

If we localize at x_i as in (0.5), we get

$$0 \longrightarrow \mathbf{I}(V)_{x_i} \longrightarrow \mathbb{C}[x_0,\ldots,x_n]_{x_i} \longrightarrow \mathbb{C}[V]_{\bar{x}_i} \longrightarrow 0$$

which is exact since localization preserves exactness (see Exercise 0.1). Taking elements of degree 0 gives the exact sequence

$$0 \longrightarrow (\mathbf{I}(V)_{x_i})_0 \longrightarrow (\mathbb{C}[x_0,\ldots,x_n]_{x_i})_0 \longrightarrow (\mathbb{C}[V]_{\bar{x}_i})_0 \longrightarrow 0.$$

Note that $(\mathbb{C}[x_0,\ldots,x_n]_{x_i})_0 = \mathbb{C}[x_0/x_i,\ldots,x_{i-1}/x_i,x_{i+1}/x_i,\ldots,x_n/x_i])$. If $f \in \mathbf{I}(V)$ is homogeneous of degree k, then

$$f/x_i^k = f(x_0/x_i,\ldots,x_{i-1}/x_i,1,x_{i+1}/x_i,\ldots,x_n/x_i) \in (\mathbf{I}(V)_{x_i})_0.$$

Thus $(\mathbf{I}(V)_{x_i})_0$ comes from the defining equations (0.4) of $V \cap U_i$. Since $\mathbb{C}[V]$ is an integral domain, the same is true for $(\mathbb{C}[V]_{\bar{x}_i})_0$. It follows that $(\mathbf{I}(V)_{x_i})_0$ is a prime ideal, which by the Nullstellensatz implies that $(\mathbf{I}(V)_{x_i})_0 = \mathbf{I}(V \cap U_i)$. The lemma follows immediately. □

One can also explore what happens when we intersect affine pieces $V \cap U_i$ and $V \cap U_i$ for $i \neq j$. By Exercise 0.2, $V \cap U_i \cap U_j$ is affine with coordinate ring

$$\mathbb{C}[V \cap U_i \cap U_j] \simeq (\mathbb{C}[V]_{x_ix_j})_0 \tag{0.7}$$

when V is irreducible. Actually, irreducibility is not essential here—(0.7) and Lemma 0.3 hold for arbitrary varieties, provided one uses the more general notion of localization mentioned in Exercise 0.2 of Chapter 1.

We will use these results in §2 when we explore the structure of projective toric varieties. We will also see later in the book that Lemma 0.3 is related to the "Proj" construction, where Proj of a graded ring gives a projective variety, just as Spec of an ordinary ring gives an affine variety.

Products of Projective Spaces. One can study the product $\mathbb{P}^n \times \mathbb{P}^m$ of projective spaces using the bigraded ring $\mathbb{C}[x_0,\ldots,x_m,y_0,\ldots,y_m]$, where x_i has bidegree $(1,0)$ and y_i has bidegree $(0,1)$. Then a bihomogeneous polynomial f of bidegree (a,b) gives a well-defined equation $f = 0$ in $\mathbb{P}^n \times \mathbb{P}^m$. This allows us to define varieties in $\mathbb{P}^n \times \mathbb{P}^m$ using bihomogeneous ideals. In particular, the ideal $\mathbf{I}(V)$ of a variety $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a bihomogeneous ideal.

Another way to study $\mathbb{P}^n \times \mathbb{P}^m$ is via the *Segre embedding*

$$\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{nm+n+m}$$

defined by mapping $(a_0, \dots, a_n, b_0, \dots, b_m)$ to the point

$$(a_0b_0, a_0b_1, \dots, a_0b_m, a_1b_0, \dots, a_1b_m, a_nb_m, a_1b_0, \dots, a_nb_m).$$

This map is studied in [**6**, Exercise 14 to Chapter 8, §4]. If $\mathbb{P}^{nm+n+m}$ has homogeneous coordinates x_{ij} for $0 \le i \le n, 0 \le j \le m$, then $\mathbb{P}^n \times \mathbb{P}^m \subseteq \mathbb{P}^{nm+n+m}$ is define by vanishing of the 2×2 minors of the $(n+1) \times (m+1)$ matrix of variables

$$\begin{pmatrix} x_{00} & \cdots & x_{0m} \\ \vdots & & \vdots \\ x_{n0} & \cdots & x_{nm} \end{pmatrix}.$$

This follows since an $(n+1) \times (m+1)$ matrix has rank 1 if and only if it is a product A^tB, where A and B are nonzero row matrices of lengths $n+1$ and $m+1$.

These approaches give the same notion of what it means to be a subvariety of $\mathbb{P}^n \times \mathbb{P}^m$. A homogeneous polynomial $F(x_{ij})$ of degree d gives the bihomogeneous polynomial $F(x_iy_j)$ of bidegree (d, d). Hence any subvariety of $\mathbb{P}^{nm+n+m}$ lying in $\mathbb{P}^n \times \mathbb{P}^m$ can be defined by a bihomogeneous ideal in $\mathbb{C}[x_0, \dots, x_m, y_0, \dots, y_m]$. Going the other way takes more thought and is discussed in Exercise 0.4.

We also have the following useful result proved in Exercise 0.5.

Proposition 0.4. *Given subvarieties $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$, the product $V \times W$ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^m$.* □

Weighted Projective Space. The graded ring associated to $\mathbb{P}^n$ is $\mathbb{C}[x_0, \dots, x_n]$, where each variable x_1 has degree 1. More generally, let $q_0, \dots, q_n$ be positive integers with $\gcd(q_0, \dots, q_n) = 1$ and define

$$\mathbb{P}(q_0, \dots, q_n) = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

where $\sim$ is the equivalence relation

$$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \iff a_i = \lambda^{q_i} b_i,\ i = 0, \dots, n \text{ for some } \lambda \in \mathbb{C}^*.$$

We call $\mathbb{P}(q_0, \dots, q_n)$ a *weighted projective space*. Note that $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$.

The ring corresponding to $\mathbb{P}(q_0, \dots, q_n)$ is the graded ring $\mathbb{C}[x_0, \dots, x_n]$ where x_i now has degree q_i. A polynomial f is *weighted homogeneous* of degree d if every monomial x^α appearing in f satisfies $\alpha \cdot (q_0, \dots, q_n) = d$. The $f = 0$ is well-defined on $\mathbb{P}(q_0, \dots, q_n)$ when f is *weighted homogeneous*, so that one can define varieties in $\mathbb{P}(q_0, \dots, q_n)$ using weighted homogeneous ideals of $\mathbb{C}[x_0, \dots, x_n]$.

Example 0.5. We can embed the weighted projective plane $\mathbb{P}(1, 1, 2)$ in $\mathbb{P}^3$ using the monomials $x_0^2, x_0x_1, x_1^2, x_2$ of weighted degree 2. In other words, the map

$$\mathbb{P}(1, 1, 2) \longrightarrow \mathbb{P}^3$$

given by

$$(a_0, a_1, a_2) \longmapsto (a_0^2, a_0a_1, a_1^2, a_2)$$

is well-defined and injective. One can check that this map induces

$$\mathbb{P}(1,1,2) \simeq \mathbf{V}(y_0y_2 - y_1^2) \subseteq \mathbb{P}^3,$$

where y_0, y_1, y_2, y_3 are homogeneous coordinates on $\mathbb{P}^3$. $\Diamond$

Later in the book we will use toric methods to construct projective embeddings of arbitrary weighted projective spaces.

Exercises for §0.

0.1. This exercise is concerned with localization over an integral domain R.

(a) Let $f \in R \setminus \{0\}$ and let $0 \to M_1 \to M_2 \to M_3$ be an exact sequence of R-modules. Prove that

$$0 \longrightarrow M_1 \otimes_R R_f \longrightarrow M_2 \otimes_R R_f \longrightarrow M_3 \otimes_R R_f \longrightarrow 0$$

is exact, where R_f is the localization of R at f defined in (0.2) of Chapter 1.

(b) Now let $R = \mathbb{C}[x_0, \ldots, x_n]$ and let $\mathbb{C}[V] = R/\mathbf{I}(V)$ for an irreducible projective variety $V \subseteq \mathbb{P}^n$. Also let $\bar{x}_i$ be the image of x_i in $\mathbb{C}[V]$. Note the $\mathbb{C}[V]$ is an R-module. Prove that $\mathbb{C}[V]_{\bar{x}_i} \simeq \mathbb{C}[V] \otimes_R R_{x_i}$.

0.2. Prove the claim made in (0.7).

0.3. Let $V \subseteq \mathbb{P}^n$ be a projective variety. Take $f_0, \ldots, f_m \in S_d$ such that the intersection $V \cap \mathbf{V}(f_0, \ldots, f_m)$ is empty. Prove that the map

$$(a_0, \ldots, a_n) \longmapsto (f_0(a_0, \ldots, a_n), \ldots, f_m(a_0, \ldots, a_n))$$

induces a well-defined map function $\Phi : V \to \mathbb{P}^m$.

0.4. Let $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be defined by $f_\ell(x_i, y_j) = 0$, where $f_\ell(x_i, y_j)$ is bihomogenous of bidegree (a_ℓ, b_ℓ), $\ell = 1, \ldots, s$. The goal of this exercise is to show that when we embed $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$ via the Segre embedding described in the text, V becomes a subvariety of $\mathbb{P}^{nm+n+m}$.

(a) For each ℓ, pick an integer $d_\ell \geq \max\{a_\ell, b_\ell\}$ and consider the polynomials $f_{\ell,\alpha,\beta} = x^\alpha y^\beta f_i(x_i, y_j)$ where $\ell = 0, \ldots, s$ and $|\alpha| = d_\ell - a_\ell$, $|\beta| = d_\ell - b_\ell$. Note that $f_{\ell,\alpha,\beta}$ is bihomogenous of bidegree (d_ℓ, d_ℓ). Prove that $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is defined by the vanishing of the $f_{\ell,\alpha,\beta}$.

(b) Use part (a) to show that V is a subvariety of $\mathbb{P}^{nm+n+m}$ under the Segre embedding.

0.5. Prove Proposition 0.4

0.6. Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Show that after relabeling coordinates, the affine cone of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ is the variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ featured in many examples in Chapter 1.

1. Lattice Points and Projective Toric Varieties

We first observe that $\mathbb{P}^n$ is a toric variety with torus

$$\begin{aligned} T_{\mathbb{P}^n} &= \mathbb{P}^n \setminus \mathbf{V}(x_0 \ldots x_n) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid a_0 \ldots a_n \neq 0\} \\ &= \{(1, t_1, \ldots, t_n) \in \mathbb{P}^n \mid t_1, \ldots, t_n \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^n. \end{aligned}$$

The action of $T_{\mathbb{P}^n}$ on itself clearly extends to an action on $\mathbb{P}^n$, making $\mathbb{P}^n$ a toric variety. To describe the lattices associated to $T_{\mathbb{P}^n}$, we use the exact sequence of tori

$$1 \longrightarrow \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^{n+1} \xrightarrow{\pi} T_{\mathbb{P}^n} \longrightarrow 1$$

coming from the definition (0.1) of $\mathbb{P}^n$. Hence the character lattice of $T_{\mathbb{P}^n}$ is

$$\mathscr{M}_n = \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \mid \textstyle\sum_{i=0}^n a_i = 0\}, \tag{1.1}$$

and the lattice of one-parameter subgroups $\mathscr{N}_n$ is the quotient

$$\mathscr{N}_n = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \dots, 1).$$

Lattice Points and Projective Toric Varieties. Let T_N be a torus with lattices M and N as usual. In Chapter 1, we used a finite set of lattice points of $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$ to create the affine toric variety $Y_{\mathscr{A}}$ as the Zariski closure of the image of the map

$$\Phi_{\mathscr{A}} : T_N \longrightarrow \mathbb{C}^s, \quad t \longmapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t)).$$

To get a projective toric variety, we regard $\Phi_{\mathscr{A}}$ as a map to $(\mathbb{C}^*)^s$ and compose with the homomorphism $\pi : (\mathbb{C}^*)^s \to T_{\mathbb{P}^{s-1}}$ to obtain

$$T_N \xrightarrow{\Phi_{\mathscr{A}}} \mathbb{C}^s \xrightarrow{\pi} T_{\mathbb{P}^{s-1}} \subseteq \mathbb{P}^{s-1}. \tag{1.2}$$

Definition 1.1. Given a finite set $\mathscr{A} \subseteq M$, the ***projective toric variety*** $X_{\mathscr{A}}$ is defined to be the Zariski closure in $\mathbb{P}^{s-1}$ of the image of the map $\Phi_{\mathscr{A}}$ from (1.2).

Proposition 1.2. *$X_{\mathscr{A}}$ is a toric variety of dimension equal to the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$.*

Proof. The proof that $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$ is a toric variety is similar to the proof given in Propostion 1.8 of Chapter 1 that $Y_{\mathscr{A}} \subseteq \mathbb{P}^s$ is a toric variety. The assertion concerning the dimension of $X_{\mathscr{A}}$ will follow from Proposition 1.6 below. □

More concretely, $X_{\mathscr{A}}$ is the Zariski closure of the image of the map

$$T_N \longrightarrow \mathbb{P}^{s-1}, \quad t \longmapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t))$$

given by the characters coming from $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$. In particular, if $M = \mathbb{Z}^n$, the χ^{m_i} is the Laurent monomial t^{m_i} and $X_{\mathscr{A}}$ as the Zariski closure of the image of

$$T_N \longrightarrow \mathbb{P}^{s-1}, \quad t \longmapsto (t^{m_1}, \dots, t^{m_s}).$$

In the literature, $\mathscr{A} \subseteq \mathbb{Z}^n$ is often given as an $n \times s$ integer matrix, so that the elements of $\mathscr{A}$ are the columns of the matrix.

Here is a example where the lattice points themselves are matrices.

Example 1.3. Let $M = \mathbb{Z}^{3\times 3}$ be the lattice of 3×3 integer matrices and let

$$\mathscr{P}_3 = \{3 \times 3 \text{ permutation matrices}\} \subseteq \mathbb{Z}^{3\times 3}.$$

Write $\mathbb{C}[M]) = \mathbb{C}[t_1^{\pm 1}, \dots, t_9^{\pm 1}]$, where the variables give the generic 3×3 matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{pmatrix}$$

with nonzero entries. Also let $\mathbb{P}^5$ have homogeneous coordinates x_{ijk} indexed by triples such that $\left(\begin{smallmatrix} 1 & 2 & 3 \\ i & j & k \end{smallmatrix}\right)$ is a permutation in S_3. Then $X_{\mathscr{P}_3} \subseteq \mathbb{P}^5$ is the Zariski closure of the image of the map $T_N \to \mathbb{P}^5$ given by the Laurent monomials $t_i t_j t_k$ for $\left(\begin{smallmatrix} 1 & 2 & 3 \\ i & j & k \end{smallmatrix}\right) \in S_3$. The ideal of $X_{\mathscr{P}_3}$ is

$$\mathbf{I}(X_{\mathscr{P}_3}) = \langle x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213}\rangle \subseteq \mathbb{C}[x_{ijk}],$$

where the relation comes from the fact that three of the permutation matrices sum to the other three (Exercise 1.1). Ideals of the toric varieties arising from permutation matrices have applications to sampling problems in statistics [**26**, p. 148]. $\diamond$

The Affine Cone of a Projective Toric Variety. The projective variety $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$ has an affine cone $\widehat{X}_{\mathscr{A}} \subseteq \mathbb{C}^s$. How does $\widehat{X}_{\mathscr{A}}$ relate to the affine toric variety $Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ constructed in Chapter 1?

Recall from Chapter 1 that when $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$, the map $e_i \mapsto m_i$ induces an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow M \tag{1.3}$$

and that the ideal of $Y_{\mathscr{A}}$ is the toric ideal

$$I_L = \big\langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \big\rangle$$

(Proposition 1.9 of Chapter 1). Then we have the following result.

Proposition 1.4. *Given $Y_{\mathscr{A}}$, $X_{\mathscr{A}}$ and I_L as above, the following are equivalent:*

(a) *$Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ is the affine cone $\widehat{X}_{\mathscr{A}}$ of $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$.*

(b) $I_L = \mathbf{I}(X_{\mathscr{A}})$.

(c) *I_L is homogeneous.*

(d) *There is $u \in N$ and $k > 0$ in $\mathbb{N}$ such that $\langle m_i, u\rangle = k$ for $i = 1, \dots, s$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from the equalities $\mathbf{I}(X_{\mathscr{A}}) = \mathbf{I}(\widehat{X}_{\mathscr{A}})$ and $I_L = \mathbf{I}(Y_{\mathscr{A}})$, and the implication (b) $\Rightarrow$ (c) is obvious.

For (c) $\Rightarrow$ (d), assume that I_L is homogeneous and take $x^\alpha - x^\beta \in L$ for $\alpha - \beta \in L$. If x^α and x^β had different degrees, then $x^\alpha, x^\beta \in I_L = \mathbf{I}(Y_{\mathscr{A}})$ would vanish on $Y_{\mathscr{A}}$. This is impossible since $(1, \dots, 1) \in Y_{\mathscr{A}}$ by (1.2). Hence x^α and

x^β have the same degree, which implies that $\ell \cdot (1, \ldots, 1) = 0$ for all $\ell \in L$. Now tensor (1.3) with $\mathbb{Q}$ and take duals to obtain an exact sequence

$$N_{\mathbb{Q}} \longrightarrow \mathbb{Q}^s \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(L_{\mathbb{Q}}, \mathbb{Q}) \longrightarrow 0.$$

The above argument shows that $(1, \ldots, 1) \in \mathbb{Q}^s$ maps to zero in $\mathrm{Hom}_{\mathbb{Q}}(L_{\mathbb{Q}}, \mathbb{Q})$ and hence comes from an element $\tilde{u} \in N_{\mathbb{Q}}$. In other words, $\langle m_i, \tilde{u} \rangle = 1$ for all i. Clearing denominators gives the desired $u \in N$ and $k > 0$ in $\mathbb{N}$.

Finally, we prove (d) $\Rightarrow$ (b). Since $I_L = \mathbf{I}(Y_{\mathscr{A}})$, it suffices to show that

$$\widehat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s \subseteq Y_{\mathscr{A}}.$$

Let $p \in \widehat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s$. Since $X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}}$ is the torus of $X_{\mathscr{A}}$, it follows that

$$p = \mu \cdot (\chi^{m_1}(t), \ldots, \chi^{m_s}(t))$$

for some $\mu \in \mathbb{C}^*$ and $t \in T_N$. Let $u \in N$ be as in part (d). This gives a one-parameter subgroup of T_N, which we write as $\tau \mapsto \lambda^u(\tau)$ for $\tau \in \mathbb{C}^*$. Then $\lambda^u(\tau)t \in T_N$ maps to the point $q \in Y_{\mathscr{A}}$ given by

$$q = \big(\chi^{m_1}(\lambda^u(\tau)t), \ldots, \chi^{m_s}(\lambda^u(\tau)t)\big) = \big(\tau^{\langle m_1, u\rangle}\chi^{m_1}(t), \ldots, \tau^{\langle m_s, u\rangle}\chi^{m_s}(t)\big),$$

since $\chi^m(\lambda^u(\tau)) = \tau^{\langle m, u\rangle}$ by the description of $\langle\ ,\ \rangle$ given in §1 of Chapter 1. The hypothesis of part (d) allows us to rewrite q as

$$q = \tau^k \cdot (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)).$$

Using $k > 0$, we can choose τ so that $p = q \in Y_{\mathscr{A}}$. This completes the proof. $\square$

The condition $\langle m_i, u \rangle = k$, $i = 1, \ldots, s$ for some $u \in N$ and $k > 0$ in $\mathbb{N}$ means that $\mathscr{A}$ lies in an affine hyperplane of $M_{\mathbb{Q}}$ not containing the origin. When $M = \mathbb{Z}^n$ and $\mathscr{A}$ consists of the columns of an $n \times s$ integer matrix, this condition is equivalent to $(1, \ldots, 1)$ lying in the row space of the matrix (Exercise 1.2).

Example 1.5. We will examine the rational normal curve $C_d \subseteq \mathbb{P}^d$ using two different sets of lattice points.

First let $\mathscr{A}$ consist of the columns of the $2 \times (d+1)$ matrix

$$A = \begin{pmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d. \end{pmatrix}$$

The columns give the Laurent monomials defining the rational normal curve C_d in Example 0.1. It follows that C_d is a projective toric variety. The ideal of C_d is the homogeneous ideal given in Example 0.1, and the corresponding affine hyperplane of $\mathbb{Z}^2$ containing $\mathscr{A}$ (= the columns of A) consists of all points (a, b) satisfying $a + b = d$. It is equally easy to see that $(1, \ldots, 1)$ is in the row space of A. In particular, we have

$$X_{\mathscr{A}} = C_d \quad \text{and} \quad Y_{\mathscr{A}} = \widehat{C}_d.$$

Now let $\mathscr{B} = \{0, 1, \ldots, d-1, d\} \subseteq \mathbb{Z}$. This gives the map

$$\Phi_{\mathscr{B}} : \mathbb{C}^* \longrightarrow \mathbb{P}^d, \quad t \mapsto (1, t, \ldots, t^{d-1}, t^d).$$

The resulting projective variety is the rational normal curve, i.e., $X_{\mathscr{B}} = C_d$, but the affine variety of $\mathscr{B}$ is *not* the rational normal cone, i.e., $Y_{\mathscr{B}} \neq \widehat{C}_d$. This can be seen by observing that $x_1^2 - x_2$ vanishes at $(1, t, \ldots, t^{d-1}, t^d) \in \mathbb{C}^{d+1}$ for all $t \in \mathbb{C}^*$. Thus $x_1^2 - x_2 \in \mathbf{I}(Y_{\mathscr{B}})$, so that $\mathbf{I}(Y_{\mathscr{B}})$ is not homogeneous. $\diamond$

Given any $\mathscr{A} \subseteq M$, there is a standard way to modify $\mathscr{A}$ so that the conditions of Proposition 1.4 are met: use $\mathscr{A} \times \{1\} \subseteq M \oplus \mathbb{Z}$. This lattice corresponds to the torus $T_n \times \mathbb{C}^*$, and since

$$\Phi_{\mathscr{A}\times\{1\}}(t, \mu) = (\chi^{m_1}(t)\mu, \ldots, \chi^{m_s}(t)\mu) = \mu \cdot (\chi^{m_1}(t), \ldots, \chi^{m_1}(t)), \tag{1.4}$$

it follows immediately that $X_{\mathscr{A}\times\{1\}} = X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$. Since $\mathscr{A} \times \{1\}$ lies in an affine hyperplane missing the origin, Proposition 1.4 implies that $X_{\mathscr{A}}$ has affine cone $Y_{\mathscr{A}\times\{1\}} = \widehat{X}_{\mathscr{A}}$. When $M = \mathbb{Z}^n$ and $\mathscr{A}$ is represented by the columns of an $n \times s$ integer matrix, we obtain $\mathscr{A} \times \{1\}$ by adding a row of 1s to the matrix.

The Torus of a Projective Toric Variety. Our next task is to determine the torus of $X_{\mathscr{A}}$. We will do so by identifying the character lattice. This will also tell us the dimension of $X_{\mathscr{A}}$. Given $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$, we set

$$\mathbb{Z}'\mathscr{A} = \big\{\textstyle\sum_{i=1}^s a_i m_i \mid a_i \in \mathbb{Z},\ \sum_{i=1}^s a_i = 0\big\}.$$

The rank of $\mathbb{Z}'\mathscr{A}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing the set $\mathscr{A}$ (Exercise 1.3).

Proposition 1.6. *Let $X_{\mathscr{A}}$ be the projective toric variety of $\mathscr{A} \subseteq M$. Then:*

(a) *The lattice $\mathbb{Z}'\mathscr{A}$ is the character lattice of torus of $X_{\mathscr{A}}$.*

(b) *The dimension of $X_{\mathscr{A}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$. In particular,*

$$\dim X_{\mathscr{A}} = \begin{cases} \operatorname{rank} \mathbb{Z}\mathscr{A} - 1 & \textit{if } Y_{\mathscr{A}} \textit{ is the affine cone of } X_{\mathscr{A}} \\ \operatorname{rank} \mathbb{Z}\mathscr{A} & \textit{otherwise.} \end{cases}$$

Proof. To prove part (a), let $M(T_{X_{\mathscr{A}}})$ be the character lattice of the torus $T_{X_{\mathscr{A}}}$ of $X_{\mathscr{A}}$. By (1.2), we have the commutative diagram

$$\begin{tikzcd} T_N \arrow[r] \arrow[dr, two heads] & \mathbb{P}^{s-1} \\ & T_{X_{\mathscr{A}}}. \arrow[u, hook] \end{tikzcd}$$

which induces the commutative diagram of character lattices

$$\begin{tikzcd} M & \mathscr{M}_{s-1} \arrow[l] \arrow[d, two heads] \\ & M(T_{X_{\mathscr{A}}}) \arrow[ul, hook] \end{tikzcd}$$

since $\mathscr{M}_{s-1} = \{(a_1,\ldots,a_s) \in \mathbb{Z}^s \mid \sum_{i=0}^s a_i m_i = 0\}$ is the character lattice of $T_{\mathbb{P}^{s-1}}$ by (1.1). The map $\mathscr{M}_{s-1} \to M$ is induced by the map $\mathbb{Z}^s \to M$ that sends e_i to m_i. Thus $\mathbb{Z}'\mathscr{A}$ is the image of $\mathscr{M}_{s-1} \to M$ and hence equals $M(T_{X_{\mathscr{A}}})$ by the above diagram.

The first assertion of part (b) follows from part (a) and the observation that $\operatorname{rank} \mathbb{Z}'\mathscr{A}$ is the dimension of the smallest affine subpace of $M_{\mathbb{R}}$ containing $\mathscr{A}$. Furthermore, if $Y_{\mathscr{A}}$ equals the affine cone of $X_{\mathscr{A}}$, then there is $u \in N$ with $\langle m_i, u\rangle = k > 0$ for all i by Proposition 1.4. Thus $\langle \sum_{i=1}^s a_i m_i, u\rangle = k(\sum_{i=1}^s a_i)$, which gives the exact sequence

$$0 \longrightarrow \mathbb{Z}'\mathscr{A} \longrightarrow \mathbb{Z}\mathscr{A} \xrightarrow{\langle\ ,u\rangle} k\mathbb{Z} \longrightarrow 0.$$

Then $k > 0$ implies $\operatorname{rank} \mathbb{Z}\mathscr{A} - 1 = \operatorname{rank} \mathbb{Z}'\mathscr{A} = \dim X_{\mathscr{A}}$. On the other hand, if $Y_{\mathscr{A}} \neq \widehat{X}_{\mathscr{A}}$, then the ideal I_L is not homogeneous, where L is defined by

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{Z}\mathscr{A} \longrightarrow 0.$$

by (1.3). Using the usual description of I_L, it follows that $m \cdot (1,\ldots,1) \neq 0$ for some $m \in L$. Hence there is $\ell > 0$ such that we have an diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & L \cap \mathscr{M}_{s-1} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \ell\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{M}_{s-1} \arrow[r] \arrow[d] & \mathbb{Z}^s \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}'\mathscr{A} \arrow[r] \arrow[d] & \mathbb{Z}\mathscr{A} \arrow[r] \arrow[d] & \mathbb{Z}/\ell\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

with exact rows and columns. Hence $\operatorname{rank} \mathbb{Z}\mathscr{A} = \operatorname{rank} \mathbb{Z}'\mathscr{A} = \dim X_{\mathscr{A}}$. $\square$

Example 1.7. Let $\mathscr{A} = \{e_1, e_2, e_1 + 2e_2, 2e_1 + e_2\} \subseteq \mathbb{Z}^2$. One computes that $\mathbb{Z}\mathscr{A} = \mathbb{Z}^2$ but $\mathbb{Z}'\mathscr{A} = \{(a,b) \in \mathbb{Z}^2 \mid a + b \equiv 0 \bmod 2\}$. Thus $\mathbb{Z}'\mathscr{A}$ has index 2 in $\mathbb{Z}\mathscr{A}$. This means that $Y_{\mathscr{A}} \neq \widehat{X}_{\mathscr{A}}$ and the map of tori

$$T_{Y_{\mathscr{A}}} \longrightarrow T_{X_{\mathscr{A}}}$$

is two-to-one, i.e., its kernel has order 2 (Exercise 1.4). $\diamond$

Affine Pieces of a Projective Toric Variety. So far, our treatment of projective toric varieties has used lattice points and toric ideals. Where are the semigroups? There are actually lots of semigroups, one for each affine piece of $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$.

The affine open set $U_i = \mathbb{P}^{s-1} \setminus \mathbf{V}(x_i)$ contains the torus $T_{\mathbb{P}^{s-1}}$. Thus

$$T_{X_{\mathscr{A}}} = X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}} \subseteq X_{\mathscr{A}} \cap U_i.$$

Since $X_{\mathscr{A}}$ is the Zariski closure of $T_{X_{\mathscr{A}}}$ in $\mathbb{P}^{s-1}$, it follows that $X_{\mathscr{A}} \cap U_i$ is the Zariski closure of $T_{X_{\mathscr{A}}}$ in $U_i \simeq \mathbb{C}^{s-1}$. Thus $X_{\mathscr{A}} \cap U_i$ is an affine toric variety.

Given $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M_{\mathbb{R}}$, the affine semigroup associated to $X_{\mathscr{A}} \cap U_i$ is easy to determine. Recall that $U_i \simeq \mathbb{C}^{s-1}$ is given by

$$(a_1, \dots, a_s) \longmapsto (a_1/a_i, \dots, a_{i-1}/a_i, a_{i+1}/a_i, \dots, a_s/a_i).$$

Combining this and $\chi^{m_j}/\chi^{m_i} = \chi^{m_j - m_i}$ with the map (1.2), we see that $X_{\mathscr{A}} \cap U_i$ is the Zariski closure of the image of the map

$$T_N \longrightarrow \mathbb{C}^{s-1}$$

given by

$$t \longmapsto \big(\chi^{m_1-m_i}(t), \dots, \chi^{m_{i-1}-m_i}(t), \chi^{m_{i+1}-m_i}(t), \dots, \chi^{m_s-m_i}(t)\big). \tag{1.5}$$

If we set $\mathscr{A}_i = \mathscr{A} - m_i = \{m_j - m_i \mid j \neq i\}$, it follows that

$$X_{\mathscr{A}} \cap U_i = X_{\mathscr{A}_i} = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i]),$$

where $\mathsf{S}_i = \mathbb{N}\mathscr{A}_i$ is the affine semigroup generated by $\mathscr{A}_i$. We have thus proved the following result.

Proposition 1.8. *Let $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$ for $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M_{\mathbb{R}}$. Then the affine piece $X_{\mathscr{A}} \cap U_i$ is the affine toric variety*

$$X_{\mathscr{A}} \cap U_i = X_{\mathscr{A}_i} = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$$

where $\mathscr{A}_i = \mathscr{A} - m_i$ and $\mathsf{S}_i = \mathbb{N}\mathscr{A}_i$. □

We also note that the results of Chapter 1 imply that the torus of $X_{\mathscr{A}_i}$ has character lattice $\mathbb{Z}\mathscr{A}_i$. Yet the torus is $T_{X_{\mathscr{A}}}$, which has character lattice $\mathbb{Z}'\mathscr{A}$ by Proposition 1.6. These are consistent since $\mathbb{Z}\mathscr{A}_i = \mathbb{Z}'\mathscr{A}$ for all i.

Besides describing the affine pieces $X_{\mathscr{A}} \cap U_i$ of $X_{\mathscr{A}} \subset \mathbb{P}^{s-1}$, we can also describe how they patch together. In other words, we can give a completely toric description of the inclusions

$$X_{\mathscr{A}} \cap U_i \;\supseteq\; X_{\mathscr{A}} \cap U_i \cap U_j \;\subseteq\; X_{\mathscr{A}} \cap U_j$$

when $i \neq j$. For instance, $U_i \cap U_j$ consists of those points of $X_{\mathscr{A}} \cap U_i$ where $x_j/x_i \neq 0$. In terms of $X_{\mathscr{A}} \cap U_i = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$, this means those points where $\chi^{m_j - m_i} \neq 0$. Thus

$$X_{\mathscr{A}} \cap U_i \cap U_j = \operatorname{Spec}\big(\mathbb{C}[\mathsf{S}_i]\big)_{\chi^{m_j-m_i}} = \operatorname{Spec}\big(\mathbb{C}[\mathsf{S}_i]_{\chi^{m_j-m_i}}\big). \tag{1.6}$$

so that if we set $m = m_j - m_i$, so that the inclusion $X_{\mathscr{A}} \cap U_i \cap U_j \subseteq X_{\mathscr{A}} \cap U_i$ can be written

$$\mathrm{Spec}(\mathbb{C}[\mathsf{S}_i])_{\chi^m} \subseteq \mathrm{Spec}(\mathbb{C}[\mathsf{S}_i]). \tag{1.7}$$

This looks very similar to the inclusion constructed in (3.4) of Chapter 1 using faces of cones. We will see in §3 that this is no accident.

We next say a few words about how the polytope $P = \mathrm{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$ relates to $X_{\mathscr{A}}$. As we will learn in §2, the dimension of P is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P, which is the same as the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$. It follows from Proposition 1.6 that

$$\dim X_{\mathscr{A}} = \dim P.$$

Furthermore, the vertices of P give an especially efficient affine covering of $X_{\mathscr{A}}$.

Proposition 1.9. *Given $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$, let $P = \mathrm{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$ and set $J = \big\{j \in \{1, \ldots, s\} \mid m_j$ is a vertex of $P\big\}$. Then*

$$X_{\mathscr{A}} = \bigcup_{j \in J} X_{\mathscr{A}} \cap U_j.$$

Proof. We will prove that if $i \in \{1, \ldots, s\}$, then $X_{\mathscr{A}} \cap U_i \subseteq X_{\mathscr{A}} \cap U_j$ for some $j \in J$. The discussion of polytopes from §2 below implies that

$$P \cap M_Q = \big\{\textstyle\sum_{j \in J} r_j m_j \mid r_j \in \mathbb{Q}_{\geq 0},\ \sum_{j \in J} r_j = 1\big\}.$$

Given $i \in \{1, \ldots, s\}$, we have $m_i \in P \cap M$, so that m_i is a convex $\mathbb{Q}$-linear combination of the vertices m_j. Clearing denominators, we get integers $k > 0$ and $k_j \geq 0$ such that

$$km_i = \textstyle\sum_{j \in J} k_j m_j, \quad \sum_{j \in J} k_j = k.$$

Thus $\sum_{j \in J} k_j(m_j - m_i) = 0$, which implies $m_i - m_j \in S_i$ when $k_j > 0$. Fix such a j. Then $\chi^{m_j - m_i} \in \mathbb{C}[\mathsf{S}_i]$ is invertible, so $\mathbb{C}[\mathsf{S}_i]_{\chi^{m_j - m_i}} = \mathbb{C}[\mathsf{S}_i]$. By (1.6), $X_{\mathscr{A}} \cap U_i \cap U_j = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_i]) = X_{\mathscr{A}} \cap U_i$, giving $X_{\mathscr{A}} \cap U_i \subseteq X_{\mathscr{A}} \cap U_j$. $\square$

We conclude with an application to projective normality. An irreducible variety $V \subseteq \mathbb{P}^n$ is called *projectively normal* if its affine cone $\widehat{V} \subseteq \mathbb{C}^{n+1}$ is normal. A projectively normal variety is always normal (Exercise 1.5). Here is an example to show that the converse can fail.

Example 1.10. Let $\mathscr{A} \subseteq \mathbb{Z}^2$ consist of the columns of the matrix

$$\begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix},$$

giving the Laurent monomials s^4, s^3t, st^3, t^4. The polytope $P = \mathrm{Conv}(\mathscr{A})$ is the line segment connecting $(4,0)$ and $(0,4)$, with vertices corresponding to s^4 and t^4. The affine piece of $X_{\mathscr{A}}$ corresponding to s^4 has coordinate ring

$$\mathbb{C}[s^3t/s^4, st^3/s^4, t^4/s^4] = \mathbb{C}[t/s, (t/s)^3, (t/s)^4] = \mathbb{C}[t/s],$$

which is normal since it is a polynomial ring. Similarly, one sees that the coordinate ring corresponding to t^4 is $\mathbb{C}[s/t]$, also normal. These affine pieces cover $X_{\mathscr{A}}$ by Proposition 1.9, so that $X_{\mathscr{A}}$ is normal.

Since $(1,1,1,1)$ is in the row space of the matrix, $Y_{\mathscr{A}}$ is the affine cone of $X_{\mathscr{A}}$ by Proposition 1.4. The affine variety $Y_{\mathscr{A}}$ is not normal by Example 3.9 of Chapter 1, so that $X_{\mathscr{A}}$ is normal but not projectively normal. $\Diamond$

The notion of normality used in this example is a bit ad-hoc since we have not formally defined normality for projective varieties. Once we define normality for abstract varieties in Chapter 3, we will see that Example 1.10 is fully rigorous.

We will say more about projective normality when we explore the connection with polytopes suggested by the above results.

Exercises for §1.

1.1. Consider the set $\mathscr{P}_3 \subseteq \mathbb{Z}^{3\times 3}$ of 3×3 permutation matrices defined in Example 1.3.

(a) Prove the claim made in Example 1.3 that three of the permutation matrices sum to the other three and use this to explain why $x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213} \in \mathbf{I}(X_{\mathscr{P}_3})$.

(b) Show that $\dim X_{\mathscr{P}_3} = 4$ by computing $\mathbb{Z}'\mathscr{P}_3$.

(c) Conclude that $\mathbf{I}(X_{\mathscr{P}_3}) = \langle x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213}\rangle$.

1.2. Let $\mathscr{A} \subseteq \mathbb{Z}^n$ consist of the columns of an $n \times s$ matrix with integer entries. Prove that the conditions of Proposition 1.4 are equivalent to the assertion that $(1,\dots,1) \in \mathbb{Z}^s$ lies in the row space of the matrix over $\mathbb{R}$ or $\mathbb{Q}$.

1.3. Given a finite set $\mathscr{A} \subseteq M$, prove that the rank of $\mathbb{Z}'\mathscr{A}$ equals the dimension of the smallest affine subspace (over $\mathbb{Q}$ or $\mathbb{R}$) containing $\mathscr{A}$.

1.4. Verify the claims made in Example 1.7. Also compute $\mathbf{I}(Y_{\mathscr{A}})$ and check that it is not homogeneous.

1.5. Let $V \subseteq \mathbb{P}^n$ be projectively normal. Use (0.6) to prove that the affine pieces $V \cap U_i$ of V are normal.

1.6. Fix a finite subset $\mathscr{A} \subseteq M$. Given $m \in M$, let $m + \mathscr{A} = \{m + m' \mid m' \in \mathscr{A}\}$. This is the *translate* of $\mathscr{A}$ by m.

(a) Prove that $\mathscr{A}$ and its translate $m + \mathscr{A}$ give the same projective toric variety, i.e., $X_{\mathscr{A}} = X_{m+\mathscr{A}}$.

(b) Give an example to show that the affine toric varieties $Y_{\mathscr{A}}$ and $Y_{m+\mathscr{A}}$ can differ. Hint: Pick $\mathscr{A}$ so that it lies in an affine hyperplane not containing the origin. Then translate $\mathscr{A}$ to the origin.

1.7. In Proposition 1.4, give a direct proof that (d) $\Rightarrow$ (c).

2. Lattice Points and Polytopes

Before we can begin our exploration of the rich connections between toric varieties and polytopes, we first need to study polytopes and their lattice points.

Polytopes. Recall from Chapter 1 that a polytope $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite set $S \subseteq M_{\mathbb{R}}$, i.e., $P = \mathrm{Conv}(S)$. Similar to what we did for cones, our discussion of polytopes will omit proofs. Detailed treatments of polytopes can be found in [**3, 11**].

The *dimension* of a polytope $P \subseteq M_{\mathbb{R}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P. Given a nonzero vector u in the dual space $N_{\mathbb{R}}$ and $b \in \mathbb{R}$, we get the *affine hyperplane* $H_{u,b}$ and *closed half-space* $H^+_{u,b}$ defined by

$$H_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle = b\} \quad \text{and} \quad H^+_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq b\}.$$

A subset $Q \subseteq P$ is a *face* of P if there are $u \in N_{\mathbb{R}} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$Q = H_{u,b} \cap P \ \text{ and } \ P \subseteq H^+_{u,b}.$$

We say that $H_{u,a}$ is a *supporting affine hyperplane* in this situation. Figure 1 shows a polygon with the supporting lines of its faces. The arrows in the figure indicate the vectors u.

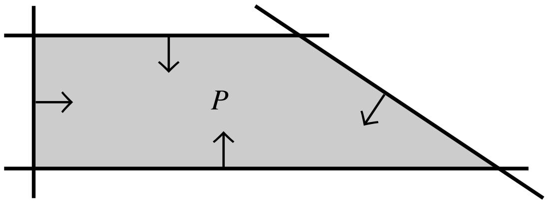


Figure 1. A polygon P and four of its supporting lines

We also regard P as a face of itself. Every face of P is again a polytope, and if $P = \mathrm{Conv}(S)$ and $Q = H_{u,b} \cap P$ as above, then $Q = \mathrm{Conv}(S \cap H_{u,b})$. Faces of P of special interest are *facets*, *edges* and *vertices*, which are faces of dimension $\dim P - 1$, 1 and 0 respectively. Facets will usually be denoted by the letter F.

Here are some properties of faces.

Proposition 2.1. *Let $P \subseteq M_{\mathbb{R}}$ be a polytope.*

(a) *P is the convex hull of its vertices.*

(b) *If $P = \mathrm{Conv}(S)$, then every vertex of P lies in S.*

(c) *If Q is a face of P, then the faces of Q are precisely the faces of P lying in Q.*

(d) *Every proper face Q is the intersection of the facets F containing Q.* □

A polytope $P \subseteq M_{\mathbb{R}}$ can also be written as a finite intersection of closed half-spaces. The converse is true provided the intersection is bounded. In other words, if an intersection

$$P = \bigcap_{i=1}^{s} H^+_{u_i,b_i}$$

is bounded, then P is a polytope. Here is a famous example.

Example 2.2. A $d \times d$ matrix $M \in \mathbb{R}^{d\times d}$ is *doubly-stochastic* if it has nonnegative entries and its row and sum columns are all 1. If we regard $\mathbb{R}^{d\times d}$ as the affine space $\mathbb{R}^{d^2}$ with coordinates a_{ij}, then the set $\mathscr{M}_d$ of all doubly-stochastic matrices is defined by the inequalites

$$\begin{array}{ll} a_{ij} \geq 0 & (\text{all } i,j) \\ \sum_{i=1}^d a_{ij} \geq 1, \;\; \sum_{i=1}^d a_{ij} \leq 1 & (\text{all } j) \\ \sum_{j=1}^d a_{ij} \geq 1, \;\; \sum_{j=1}^d a_{ij} \leq 1 & (\text{all } i). \end{array}$$

(We use two inequalities to get one equality.) These inequalities easily imply that $0 \leq a_{ij} \leq 1$ for all i,j, so that $\mathscr{M}_d$ is bounded and hence is a polytope.

Birkhoff and Von Neumann proved independently that the vertices of $\mathscr{M}_d$ are the $d!$ permutation matrices and that $\dim \mathscr{M}_d = (d-1)^2$. In the literature, $\mathscr{M}_d$ has various names, including the *Birkhoff polytope* and the *transportation polytope*. See [**28**, p. 20] for more on this interesting polytope. $\Diamond$

When P has maximal dimension, i.e., $\dim P = \dim M_{\mathbb{R}}$, its presentation as an intersection of closed half-spaces has an especially nice form because each facet F has a *unique* supporting affine hyperplane. We write the supporting affine hyperplane and corresponding closed half-space as

$$H_F = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle = -a_F\} \text{ and } H_F^+ = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F\},$$

where $(u_F, a_F) \in N_{\mathbb{R}} \times \mathbb{R}$ is unique up to multiplication by a positive real number. We call u_F an *inward-pointing facet normal* of the facet F. It follows that

$$(2.1) \qquad P = \bigcap_{F \text{ facet}} H_F^+ = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F \text{ of } P\}.$$

In Figure 1, the supporting lines plus arrows determine the supporting half-planes whose intersection is the polygon P. We write (2.1) with minus signs in order to simplify formulas in later chapters.

Here are some important classes of polytopes.

Definition 2.3. Let $P \subseteq M_{\mathbb{R}}$ be a polytope of dimension d.

(a) P is a ***simplex*** or ***d-simplex*** if it has $d+1$ vertices.

(b) P is ***simplicial*** if every facet of P is simplicial.

(c) P is ***simple*** if every vertex is the intersection of precisely d facets.

Examples include the Platonic solids in $\mathbb{R}^3$:

- A tetrahedron is a 3-simplex.
- The octahedron and icosahedron are simplicial since their facets are triangles.
- The cube and dodecahedron are simple since three facets meet at every vertex.

Polytopes P_1 and P_2 are *combinatorially equivalent* if there is a bijection

$$\{\text{faces of } P_1\} \simeq \{\text{faces of } P_2\}$$

that preserves dimensions, inclusions and intersections. For example, simplices of the same dimension are combinatorially equivalent, and in the plane, the same holds for polygons with the same number of vertices.

Sums, Multiples, and Duals. Given a polytope $P = \mathrm{Conv}(S)$, its multiple $rP = \mathrm{Conv}(rS)$ is again a polytope for any $r \geq 0$. If P is defined by the inequalities

$$\langle m, u_i \rangle \geq a_i, \quad 1 \leq i \leq s,$$

then rP is given by

$$\langle m, u_i \rangle \geq ra_i, \quad 1 \leq i \leq s.$$

In particular, when P has maximal dimension, then P and rP have the same inward-pointing facet normals.

The *Minkowski sum* of subsets $A_1, A_2 \subseteq M_{\mathbb{R}}$ is

$$A_1 + A_1 = \{a_1 + a_2 \mid a_1 \in A_1, a_2 \in A_2\}.$$

Given polytopes $P_1 = \mathrm{Conv}(S_1)$ and $P_1 = \mathrm{Conv}(S_2)$, their Minkowski sum $P_1 + P_2 = \mathrm{Conv}(S_1 + S_2)$ is again a polytope. We also have the distributive law

$$rP + sP = (r + s)P.$$

When P has maximal dimension and 0 is an interior point of P, we define the *dual* or *polar* polytope

$$P^\circ = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq -1 \text{ for all } u \in P\}.$$

Figure 2 shows an example of this in the plane.

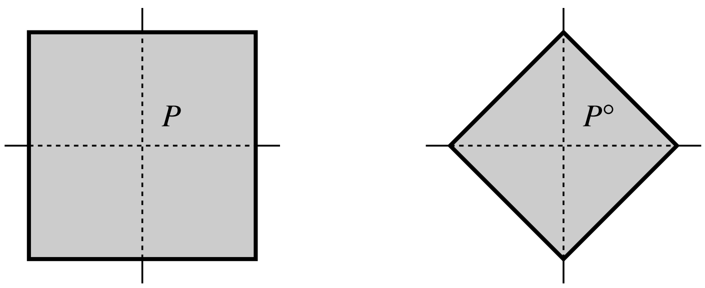


Figure 2. A polygon P and its dual P° in the plane

When we write $P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F, F \text{ facet}\}$ as in (2.1), we have $a_F > 0$ for all F since 0 is in the interior. Then P° is the convex hull of the vectors $(1/a_F)u_F \in N_{\mathbb{R}}$ (Exercise 2.1). We also have $(P^\circ)^\circ = P$ in this situation.

Lattice Polytopes. Now let M and N be dual lattices with associated vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. A ***lattice polytope*** $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite set $S \subseteq M$. It follows easily that a polytope in $M_{\mathbb{R}}$ is a lattice polytope if and only if its vertices lie in M (Exercise 2.2).

Example 2.4. The *standard n-simplex* in $\mathbb{R}^n$ is

$$\Delta_n = \mathrm{Conv}(0, e_1, \ldots, e_n).$$

Another simplex in $\mathbb{R}^3$ is $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$, shown in Figure 3.

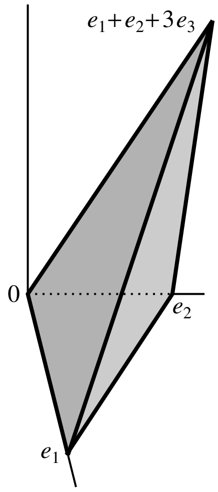


Figure 3. The simplex $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$

The lattice polytopes Δ_3 and P are combinatorially equivalent but will give very different projective toric varieties. $\Diamond$

Example 2.5. The Birkhoff polytope defined in Example 2.2 is a lattice polytope relative to the lattice of integer matrices $\mathbb{Z}^{d \times d}$ since its vertices are the permutation matrices, whose entries are all 0 or 1. $\Diamond$

One can show that faces of lattice polytopes are again lattice polytopes and that Minkowski sums and integer multiples of lattice polytopes are lattice polytopes (Exercise 2.2). Furthermore, every lattice polytope is an intersection of closed half-spaces defined over M, i.e., $P = \bigcap_{i=1}^s H_{u_i,b_i}^+$ where $u_i \in N$ and $b_i \in \mathbb{Z}$.

When a lattice polytope P has maximal dimension, the facet presentation given in (2.1) has an especially nice form. If F is a facet of P, then the inward-pointing facet normals of F lie on a rational ray in $N_{\mathbb{R}}$. Let u_F denote the unique ray generator. The corresponding a_F is integral since $\langle m, u_F\rangle = -a_F$ when m is a vertex of F. It follows that

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F \text{ of } P\}. \tag{2.2}$$

is the *unique* facet presentation of the lattice polytope P.

Example 2.6. Conisder the square $P = \mathrm{Conv}(\pm e_1 \pm e_2) \subseteq \mathbb{R}^2$. The facet normals of P are $\pm e_1$ and $\pm e_2$ and the facet presentation of P is given by the inequalities

$$\begin{aligned}\langle m, \pm e_1\rangle &\geq -1\\ \langle m, \pm e_2\rangle &\geq -1.\end{aligned}$$

Since the a_F are all equal to 1, it follows that $P^\circ = \mathrm{Conv}(\pm e_1, \pm e_2)$ is also a lattice polytope. The polytopes P and P° are pictured in Figure 2.

It is rare that the dual of a lattice polytope is a lattice polytope—this is related to the *reflexive polytopes* that will be studied later in the book.

Example 2.7. The 3-simplex $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ pictured in Example 2.4 has facet presentation

$$\begin{aligned}\langle m, e_3\rangle &\geq 0\\ \langle m, 3e_1 - e_3\rangle &\geq 0\\ \langle m, 3e_2 - e_3\rangle &\geq 0\\ \langle m, -3e_1 - 3e_2 + e_3\rangle &\geq -3\end{aligned}$$

(Exercise 2.3). However, if we replace -3 with -1 in the last inequality, we get integer inequalities that define $(1/3)P$, which is *not* a lattice polytope. $\Diamond$

The combinatorial type of a polytope is an interesting object of study. This leads to the question "Is every polytope combinatorially equivalent to a lattice polytope?" If given polytope is simplicial, the answer is "yes"—just wiggle the vertices to make them rational and clear denominators to get a lattice polytope. The same argument works for simple polytopes by wiggling the facet normals. This will enable us to prove results about arbitrary simplicial or simple polytopes using toric varieties. But in general, the answer is "no"—there exist polytopes in every dimension ≥ 8 not combinatorially equivalent to any lattice polytope. An example is described in [**28**, Example 6.21].

Normal Polytopes. The connection between lattice polytopes and toric varieties comes from the lattice points of the polytope. Unfortunately, a lattice polytope might not have enough lattice points. The 3-simplex P from Example 2.7 has only four lattice points (its vertices), which implies that the projective toric variety $X_{P\cap\mathbb{Z}^3}$ is just $\mathbb{P}^3$ (Exercise 2.3).

We will explore two notions of what it means for a lattice polytope to "have enough lattice points." Here is the first.

Definition 2.8. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is ***normal*** if

$$(kP) \cap M + (\ell P) \cap M = ((k+\ell)P) \cap M.$$

for all $k, \ell \in \mathbb{N}$.

The inclusion $(kP) \cap M + (\ell P) \cap M \subseteq ((k+\ell)P) \cap M$ is automatic. Thus normality means that all lattice points of $(k+\ell)P$ come from lattice points of kP and ℓP. In particular, a lattice polytope is normal if and only if

$$\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} = (kP) \cap M.$$

for all integers $k \geq 1$. In other words, normality says that P has enough lattice points to generate the lattice points in all integer multiples of P.

Definition 2.9. A simplex $P \subseteq M_{\mathbb{R}}$ is ***basic*** if P has a vertex m_0 such that the differences $m - m_0$, for vertices $m \neq m_0$ of P, form a subset of a $\mathbb{Z}$-basis of M.

This definition is independent of which vertex $m_0 \in P$ is chosen. The standard simplex $\Delta_n \subseteq \mathbb{R}^n$ is basic, and any basic simplex is normal (Exercise 2.4).

Here is another interesting class of normal polytopes.

Proposition 2.10. *Every lattice polygon $P \subseteq \mathbb{R}^2$ is normal.*

Proof. In Exercise 2.5 you will show that P is a union of lattice triangles $P = \bigcup_i T_i$ such that each T_i has only vertices as lattice points. Such lattice triangles are basic by Exercise 2.6. Take a lattice point p of $(k+\ell)P$. This lies in $(k+\ell)T_i$ for some i. The basic triangle T_i is normal, so that p lies in

$$(kT_i) \cap \mathbb{Z}^2 + (\ell T_i) \cap \mathbb{Z}^2 \subseteq (kP) \cap \mathbb{Z}^2 + (\ell P) \cap \mathbb{Z}^2. \qquad \square$$

The property "lattice triangles with only vertices as lattice points are basic and hence normal" fails in higher dimensions.

Example 2.11. Let $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$. We noted earlier that the only lattice points of P are its vertices. But P is not normal since

$$e_1 + e_2 + e_3 = \tfrac{1}{6}(0) + \tfrac{1}{3}(2e_1) + \tfrac{1}{3}(2e_2) + \tfrac{1}{6}(2e_1 + 2e_2 + 6e_3) \in 2P$$

is not the sum of lattice points of P. It follows that P is not basic. $\Diamond$

We can also interpret normality in terms of the cone

$$C_P = \mathrm{Cone}(P \times \{1\}) \subseteq M_{\mathbb{R}} \times \mathbb{R}$$

introduced in Figure 3 of Chapter 1. The key feature of this cone is that kP is the "slice" of C_P at height k, as illustrated in Figure 4 on the next page. It follows that lattice points $m \in kP$ correspond to points $(m, k) \in C_P \cap (M \times \mathbb{Z})$.

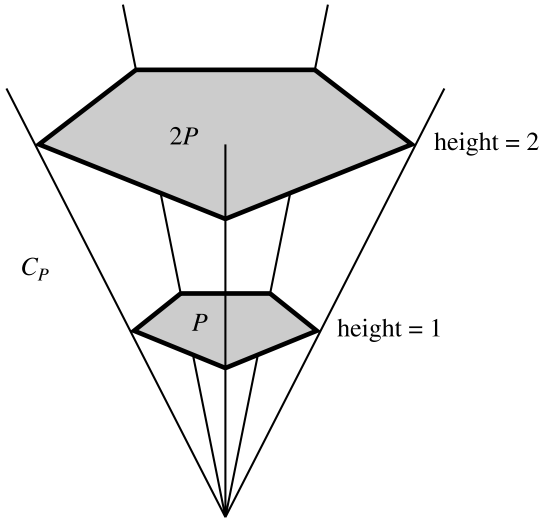


Figure 4. The cone C_P sliced at heights 1 and 2

In Exercise 2.7 you will show that the semigroup $C_P \cap (M \times \mathbb{Z})$ of lattice points in C_P relates to normality as follows.

Lemma 2.12. *The lattice polytope $P \subseteq M_{\mathbb{R}}$ is normal if and only if $(P \cap M) \times \{1\}$ generates the semigroup $C_P \cap (M \times \mathbb{Z})$.* □

Here is an important result on normality.

Theorem 2.13. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope. Then there is an integer $k_0 > 0$ such that kP is normal for all $k \geq k_0$.*

Proof. Let $\mathscr{H} = \{(m_i, h_i)\}, i = 1, \ldots, s$, be the Hilbert basis of $C_P \cap (M \times \mathbb{Z})$. Note that $m_i \in h_i P$. In Exercise 2.8 you will adapt the proof of Gordan's Lemma (Proposition 2.15 from Chapter 1) to show that if $\mathscr{H}$ is the Hilbert basis of the semigroup of lattice points in a strongly convex cone $\mathrm{Cone}(\mathscr{A})$, then the lattice points in the cone can be written as the union

$$\mathbb{N}\mathscr{A} \cup \textstyle\bigcup_{m \in \mathscr{H}} \big(m + \mathbb{N}\mathscr{A}\big). \tag{2.3}$$

Applied to $\mathscr{A} = (P \cap M) \times \{1\} \subseteq M \times \mathbb{Z}$, this implies that

$$C_P \cap (M \times \mathbb{Z}) = \mathsf{S} \cup \textstyle\bigcup_{i=1}^{s} \big((m_i, h_i) + \mathsf{S}\big), \tag{2.4}$$

where $\mathsf{S} = \mathbb{N}((P \cap M) \times \{1\})$.

Now let $k_0 = \max\{h_i \mid i = 1, \ldots, s\}$ and take $k \geq k_0$. We will prove that kP is normal by showing that

$$(kP) \cap M + (dkP) \cap M = ((d+1)kP) \cap M$$

for all $d \in \mathbb{N}$. It suffices to prove $\supseteq$, so let $m \in ((d+1)kP) \cap M$. Then $(m,(d+1)k) \in C_P \cap (M \times \mathbb{Z})$, which means that $(m,(d+1)k)$ lies in the union (2.4). If $(m,(d+1)k) \in \mathsf{S}$, then m is a sum of $(d+1)k$ points from $P \cap M$. Writing this as a sum of k points plus a sum of dk points shows that $m \in (kP) \cap M + (dkP) \cap M$. Also, if $(m,(d+1)k) \in (m_i,h_i) + \mathsf{S}$, then

$$m = m_i + \big(\text{sum of } (d+1)k - h_i \text{ points of } P \cap M\big).$$

Since $k \geq h_i$, we can write this as

$$m = \big(m_i + \text{sum of } k - h_i \text{ points of } P \cap M\big) + \big(\text{sum of } dk \text{ points of } P \cap M\big).$$

Since $m_i \in h_i P$, it follows that $m \in (kP) \cap M + (dkP) \cap M$. □

Here is an example of this result.

Example 2.14. Consider $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$ from Example 2.11. The Hilbert basis of $C_P \cap (M \times \mathbb{Z})$ consists of the six points

$$(0,1), (e_1,1), (e_2,1), (e_1+e_2+3e_3,1), (e_1+e_2+e_3,2)(e_1+e_2+2e_3,2)$$

(Exercise 2.3). It follows from the proof of Theorem 2.13 that $2P$ is normal.

In Chapter 6 we will sharpen Theorem 2.13 by showing that if P is an n-dimensional lattice polytope, then kP is normal for all $k \geq n-1$. This was first proved in [**5**] and explains why $2P$ is normal in Example 2.14. Also, in Chapter 7 we will show that the number of lattice points in nP is given by a polynomial in n called the *Ehrhart polynomial*.

Very Ample Polytopes. Here is a slightly different notion of what it means for a polytope to have enough lattice points.

Definition 2.15. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is ***very ample*** if for every vertex $m \in P$, the semigroup $\mathsf{S}_{P,m} = \mathbb{N}(P \cap M - m)$ generated by $P \cap M - m = \{m' - m \mid m' \in P \cap M\}$ is saturated in M.

This definition relates to normal polytopes as follows.

Proposition 2.16. *A normal lattice polytope P is very ample.*

Proof. Fix a vertex $m_0 \in P$ and take $m \in M$ such that $km \in \mathsf{S}_{P,m_0}$ for some integer $k \geq 1$. To prove that $m \in \mathsf{S}_{P,m_0}$, write $km \in \mathsf{S}_{P,m_0}$ as

$$km = \textstyle\sum_{m' \in P \cap M} a_{m'}(m' - m_0), \quad a_{m'} \in \mathbb{N}.$$

Pick $d \in \mathbb{N}$ so that $kd \geq \sum_{m' \in P \cap M} a_{m'}$. Then

$$km + kdm_0 = \textstyle\sum_{m' \in P \cap M} a_{m'} m' + \big(kd - \sum_{m' \in P \cap M} a_{m'}\big) m_0 \in kdP.$$

Dividing by k gives $m + dm_0 \in dP$, which by normality implies that

$$m + dm_0 = \sum_{i=1}^{d} m_i, \quad m_i \in P \cap M \text{ for all } i.$$

We conclude that $m = \sum_{i=1}^{d}(m_i - m_0) \in \mathsf{S}_{P,m_0}$, as desired. □

Combining this with Proposition 2.10 and Theorem 2.13 gives the following.

Corollary 2.17. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope.*

(a) *If $\dim P = 2$, then P is very ample.*

(b) *There is an integer $k_0 \geq 1$ such that kP is very ample for all $k \geq k_0$.* □

We will see in the next section that very ampleness is precisely the property we need in order to define the toric variety of a lattice polytope. Also, the sharpening of Theorem 2.13 mentioned earlier will enable us to take $k_0 = \dim P - 1$ in part (b) of the corollary.

The following example taken from [**4**, Example 5.1] shows that very ample polytopes need not be normal, i.e., the converse of Proposition 2.16 is false.

Example 2.18. Given $1 \leq i < j < k \leq 6$, let $[ijk]$ denote the vector in $\mathbb{Z}^6$ with 1 in positions i, j, k and 0 elsewhere. Thus $[123] = (1,1,1,0,0,0)$. Then let

$$\mathscr{A} = \big\{[123],[124],[135],[146],[156],[236],[245],[256],[345],[346]\big\} \subseteq \mathbb{Z}^6.$$

The lattice polytope $P = \mathrm{Conv}(\mathscr{A})$ lies in the affine hyperplane of $\mathbb{R}^6$ where the coordinates sum to 3. As explained in [**4**], this configuration can be interpreted in terms of a triangulation of the real projective plane.

The points of $\mathscr{A}$ of P are the only lattice points of P (Exercise 2.9), so that $\mathscr{A}$ is the set of vertices of P. Number the points of $\mathscr{A}$ as $m_1, \ldots, m_{10}$. Then

$$(1,1,1,1,1,1) = \tfrac{1}{5}\sum_{i=1}^{10} m_i = \sum_{i=1}^{10} \tfrac{1}{10}(2m_i)$$

shows that $v = (1,1,1,1,1,1) \in 2P$. Since v is not a sum of lattice points of P (when $[ijk] \in \mathscr{A}$, the vector with 1s in the complementary positions is not in $\mathscr{A}$), we conclude that P is not normal.

Showing that P is very ample takes more work. The first step is to prove that $\mathscr{A} \times \{1\} \cup \{(v,2)\} \subseteq \mathbb{R}^6 \times \mathbb{R}$ is a Hilbert basis of the semigroup $C_P \cap \mathbb{Z}^7$, where $C_P \subseteq \mathbb{R}^6 \times \mathbb{R} = \mathbb{R}^7$ is the cone over $P \times \{1\}$. We used the software 4ti2 [**14**].

Now fix i and let S_{P,m_i} be the semigroup generated by the $m_j - m_i$. Take $m \in \mathbb{Z}^6$ such that $km \in \mathsf{S}_{P,m_i}$. As in the proof of Proposition 2.16, this implies that $m + dm_i \in dP$ for some $d \in \mathbb{N}$. Thus $(m + dm_i, d) \in C_P \cap \mathbb{Z}^7$. Expressing this in terms of the above Hilbert basis easily implies that

$$m = a(v - 2m_i) + \sum_{j=1}^{10} a_j(m_j - m_i), \quad a, a_j \in \mathbb{N}.$$

If we can show that $v - 2m_i \in \mathsf{S}_{P,m_i}$, then $m \in \mathsf{S}_{P,m_i}$ follows immediately and proves that S_{P,m_i} is saturated. When $i = 1$, one can check that

$$v + [123] = [124] + [135] + [236],$$

which implies that

$$v - 2m_1 = (m_2 - m_1) + (m_3 - m_1) + (m_6 - m_1) \in \mathsf{S}_{P,m_1}.$$

One obtains similar formulas for $i = 2, \ldots, 10$ (Exercise 2.9), which completes the proof that P is very ample.

The polytope P has further interesting properties. For example, up to affine equivalence, P can be described as the convex hull of the 10 points in $\mathbb{Z}^5$ given by

$$\begin{gathered}(0,0,0,0,0), (0,0,0,0,1), (0,0,1,1,0), (0,1,0,1,1), (0,1,1,1,0)\\ (1,0,1,0,1), (1,0,1,1,1), (1,1,0,0,0), (1,1,0,1,1), (1,1,1,0,0).\end{gathered}$$

Of all 5-dimensional polytopes whose vertices lie in $\{0,1\}^5$, this polytope has the most edges, namely 45 (see [**1**]). Since it has 10 vertices and $45 = \binom{10}{2}$, every pair of distinct vertices is joined by an edge. Such polytopes are *2-neighborly*. ◇

Exercises for §2.

2.1. Let $P \subseteq M_{\mathbb{R}}$ be a polytope of maximal dimension with the origin as an interior point.

(a) Write $P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F$ for all facets $F\}$. Prove that $a_F > 0$ for all F and that $P^\circ = \mathrm{Conv}((1/a_F)u_F \mid F$ a facet$)$.

(b) Prove that the dual of a simplicial polytope is simple and vice versa.

(c) Prove that $(rP)^\circ = (1/r)P^\circ$ for all $r > 0$.

(d) Use part (c) to construct an example of a lattice polytope whose dual is not a lattice polytope.

2.2. Let $P \subseteq M_{\mathbb{R}}$ be a polytope.

(a) Prove that P is a lattice polytope if and only if the vertices of P lie in M

(b) Prove that P is a lattice polytope if and only if P is the convex hull of its lattice points, i.e., $P = \mathrm{Conv}(P \cap M)$.

(c) Prove that every face of a lattice polytope is a lattice polytope.

(d) Prove that Minkowski sums and integer multiples of lattice polytopes are again lattice polytopes.

2.3. Let $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ be the simplex studied in Examples 2.4, 2.7, 2.11 and 2.14.

(a) Verify the facet presentation of P given in Example 2.7.

(b) Show that the only lattice points of P are its vertices.

(c) Show that the toric variety $X_{P\cap\mathbb{Z}^3}$ is $\mathbb{P}^3$.

(d) Show that the vectors given in Example 2.14 form the Hilbert basis of $C_P \cap (M \times \mathbb{Z})$.

2.4. Recall the definition of basic simplex given in Definition 2.9.

(a) Show that if a simplex satisfies Definition 2.9 for one vertex, then it satisfies the definition for all vertices.

(b) Show that the standard simplex Δ_n is basic.

(c) Prove that a basic simplex is normal.

2.5. Let P be a lattice polygon. Prove that P can be written as a union of lattice triangles such that each lattice triangle has only vertices as lattice points. Hint: First use induction on the number of vertices to reduce to the case when P is a lattice triangle. Then use induction on the number of lattice points. This will construct a *triangulation* of P.

2.6. Prove that a lattice triangle with only vertices as lattice points is basic. Hint: Translate one vertex to the origin and consider the cone generated by the other two vertices.

2.7. Prove Lemma 2.12.

2.8. Prove (2.3).

2.9. Consider the polytope $P = \mathrm{Conv}(\mathscr{A})$ from Example 2.18.

(a) Prove that $\mathscr{A}$ is the set of lattice points of P.

(b) Complete the proof begun in the text that P is very ample.

2.10. Prove that every proper face of a simplicial polytope is a simplex.

2.11. In Corollary 2.17 we proved that kP is very ample for k sufficiently large using Theorem 2.13 and Proposition 2.16. Give a direct proof of this result. Hint: A vertex $m \in P$ gives the cone $C_{P,m}$ generated by the semigroup $\mathsf{S}_{P,m}$ defined in Definition 2.15. The cone $C_{P,m}$ is strongly convex since m is a vertex and hence $C_{P,m} \cap M$ has a Hilbert basis. Furthermore, $C_{P,m} = C_{kP,km}$ for all $k \in \mathbb{N}$. Now argue that when k is large, $(kP) \cap M - km$ contains the Hilbert basis of $C_{P,m} \cap M$. A picture will help.

2.12. Fix an integer $a \geq 1$ and consider the 3-simplex $P = \mathrm{Conv}(0, ae_1, ae_2, e_3) \subseteq \mathbb{R}^3$.

(a) Work out the facet presentation of P and verify that the facet normals can be labeled so that $u_0 + u_1 + u_2 + au_3 = 0$.

(b) Show that P is normal. Hint: Show that $P \cap \mathbb{Z}^3 + (kP) \cap \mathbb{Z}^3 = ((k+1)P) \cap \mathbb{Z}^3$.

We will see later that the toric variety of P is the weighted projective space $\mathbb{P}(1,1,1,a)$.

3. Polytopes and Projective Toric Varieties

Our next task is to define the toric variety of a lattice polytope. As noted in §2, we need to make sure that the polytope has enough lattice points. Hence we begin with very ample polytopes. Strongly convex rational polyhedral cones will play an important role in our development.

The Very Ample Case. Let $P \subseteq M_{\mathbb{R}}$ be a very ample polytope relative to the lattice M, and assume that $\dim P = \dim M_{\mathbb{R}} = \operatorname{rank} M$. We define

$$X_P = X_{P\cap M}.$$

To see what this means concretely, write $P \cap M = \{m_1, \dots, m_s\}$. Then X_P is the Zariski closure of the image of the map $T_N \to \mathbb{P}^{s-1}$ given by

$$t \longmapsto \big(\chi^{m_1}(t), \dots, \chi^{m_s}(t)\big) \in \mathbb{P}^{s-1}.$$

Fix homogeneous coordinates $x_1, \dots, x_s$ for $\mathbb{P}^{s-1}$.

We examine the structure of $X_P \subseteq \mathbb{P}^{s-1}$ using Propositions 1.8 and 1.9. For each $m_i \in P \cap M$ consider the semigroup

$$\mathsf{S}_i = \mathbb{N}(P \cap M - m_i)$$

generated by $m_j - m_i$ for $m_j \in P \cap M$. In $\mathbb{P}^{s-1}$ we have the affine open subset $U_i \simeq \mathbb{C}^{s-1}$ consisting of those points where $x_i \neq 0$. Proposition 1.8 showed that the affine open piece $X_P \cap U_i$ of X_P is the affine toric variety

$$X_P \cap U_i \simeq \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i]),$$

and Proposition 1.9 showed that

$$X_P = \bigcup_{m_i \text{ vertex of } P} X_P \cap U_i.$$

Here is our first major result about X_P.

Theorem 3.1. *Let X_P be the projective toric variety of the very ample polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$, and assume that* $\dim P = n$.

(a) *For each vertex $m_i \in P \cap M$, the affine piece $X_P \cap U_i$ is the affine toric variety*

$$X_P \cap U_i = U_{\sigma_i} = \operatorname{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$$

where $\sigma_i \subseteq N_{\mathbb{R}}$ is the strongly convex rational polyhedral cone dual to the cone $\operatorname{Cone}(P \cap M - m_i) \subseteq M_{\mathbb{R}}$. *Furthermore,* $\dim \sigma_i = n$.

(b) *The torus of X_P has character lattice M and hence is the torus T_N.*

Proof. Let $C_i = \operatorname{Cone}(P \cap M - m_i)$. Since m_i is a vertex, it has a supporting hyperplane $H_{u,a}$ such that $P \subseteq H_{u,a}^+$ and $P \cap H_{u,a} = \{m_i\}$. It follows that $H_{u,0}$ is a supporting hyperplane of $0 \in C_i$ (Exericse 3.1), so that C_i is strongly convex. It is also easy to see that $\dim C_i = \dim P$ (Exericse 3.1). It follows that C_i and $\sigma_i = C_i^\vee$ are strongly convex rational polyhedral cones of dimension n.

We have $\mathsf{S}_i \subseteq C_i \cap M = \sigma_i^\vee \cap M$. By hypothesis, P is very ample, which means that $\mathsf{S}_i \subseteq M$ is saturated. Since S_i and $C_i = \sigma_i^\vee$ are both generated by $P \cap M - m_i$, part (a) of Exercise 3.4 of Chapter 1 implies $\mathsf{S}_i = \sigma_i^\vee \cap M$. (This exercise was part of the proof of the characterization of normal affine toric vareities given in Theorem 3.5 of Chapter 1.) Part (a) of the theorem follows immediately.

For part (b), Theorem 2.16 implies that T_N is the torus of U_{σ_i} since σ_i is strongly convex. Then $T_N \subseteq U_{\sigma_i} = X_P \cap U_i \subseteq X_P$ shows that T_N is also the torus of X_P. □

The affine pieces $X_P \cap U_i$ and $X_P \cap U_j$ intersect in $X_P \cap U_i \cap U_j$. In order to describe this intersection carefully, we need to study how the cones σ_i and σ_j fit together in $N_{\mathbb{R}}$. This leads to our next topic.

The Normal Fan. The cones $\sigma_i \subseteq N_{\mathbb{R}}$ appearing in Theorem 3.1 fit together in a remarkably nice way, giving a structure called the *normal fan of* P.

Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a lattice polytope of dimension n, not necessarily very ample. Faces, facets and vertices of P will be denoted by Q, F and v respectively. Hence we write the facet presentation of P as

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all } F\}. \tag{3.1}$$

A vertex $v \in P$ gives cones

$$C_v = \mathrm{Cone}(P \cap M - v) \subseteq M_{\mathbb{R}} \quad \text{and} \quad \sigma_v = C_v^\vee \subseteq N_{\mathbb{R}}.$$

(When $v = m_i$, these are the cones C_i and σ_i studied above.) Faces $Q \subseteq P$ containing v correspond bijectively to faces $\boldsymbol{Q} \subseteq C_v$ via the maps

$$\begin{aligned} Q &\longmapsto \boldsymbol{Q} = \mathrm{Cone}(Q \cap M - v) \\ \boldsymbol{Q} &\longmapsto Q = (\boldsymbol{Q} + v) \cap P \end{aligned} \tag{3.2}$$

which are inverses of each other. These maps preserve dimensions, inclusions, and intersections (Exercise 3.2), as illustrated in Figure 5.

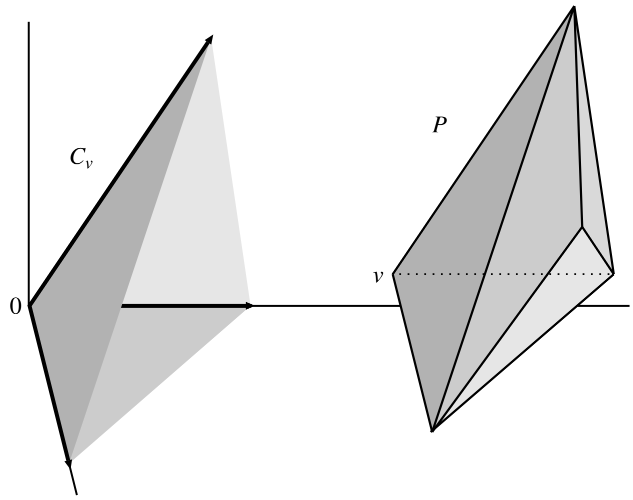


Figure 5. The cone C_v of a vertex $v \in P$

In particular, all facets of C_v come from facets of P containing v, so that

$$C_v = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq 0 \text{ for all } F \text{ containing } v\}.$$

By the duality results of Chapter 1, it follows that the dual cone σ_v is given by

$$\sigma_v = \mathrm{Cone}(u_F \mid F \text{ contains } v).$$

This construction generalizes to arbitrary faces $Q \subseteq P$ by setting

$$\sigma_Q = \mathrm{Cone}(u_F \mid F \text{ contains } Q).$$

In particular, σ_F is the ray generated by u_F, and $\sigma_P = \{0\}$ since the $\{0\}$ is the cone generated by the empty set.

The following easy lemma will be useful.

Lemma 3.2. *Let Q be a face of P and let $H_{u,b}$ be a supporting affine hyperplane of P. Then $u \in \sigma_Q$ if and only if $Q \subseteq H_{u,b} \cap P$.*

Proof. First suppose that $u \in \sigma_Q$ and write $u = \sum_{Q\subseteq F} \lambda_F u_F$, $\lambda_F \geq 0$. Then setting $b = -\sum_{Q\subseteq F} \lambda_F a_F$ easily implies that $P \subseteq H_{u,b}^+$ and $Q \subseteq H_{u,b} \cap P$. Recall that the integers a_F come from the facet presentation (3.1).

Going the other way, suppose that $Q \subseteq H_{u,b} \cap P$. Take a vertex $v \in Q$. Then $P \subseteq H_{u,b}^+$ and $v \in H_{u,b}$ imply that $C_v \subseteq H_{u,0}^+$. Hence $u \in C_v^\vee = \sigma_v$, so that

$$u = \textstyle\sum_{v\in F} \lambda_F u_F, \quad \lambda_F \geq 0.$$

Let F_0 be a facet of P containing v but not Q, and pick $p \in Q$ with $p \notin F_0$. Then $p, v \in Q \subseteq H_{u,b}$ imply that

$$\begin{aligned} b &= \langle p, u\rangle = \textstyle\sum_{v\in F} \lambda_F \langle p, u_F\rangle \\ b &= \langle v, u\rangle = \textstyle\sum_{v\in F} \lambda_F \langle v, u_F\rangle = -\textstyle\sum_{v\in F} \lambda_F a_F. \end{aligned}$$

where the last equality uses $\langle v, u_F\rangle = -a_F$ for $v \in F$. These equations imply

$$\textstyle\sum_{v\in F} \lambda_F \langle p, u_F\rangle = -\textstyle\sum_{v\in F} \lambda_F a_F.$$

However, $p \notin F_0$ gives $\langle p, u_{F_0}\rangle > -a_{F_0}$, and since $\langle p, u_F\rangle \geq -a_F$ for all F, it follows that $\lambda_{F_0} = 0$ whenever $Q \not\subseteq F_0$. This gives $u \in \sigma_Q$ and completes the proof of the lemma. □

Corollary 3.3. *If Q is a face of F, then $u_F \in \sigma_Q$ if and only if $Q \subseteq F$.*

Proof. One direction is obvious by the definition of σ_Q, and the other direction follows from Lemma 3.2 since $H_{u_F,-a_F}$ is a supporting affine hyperplane of P with $H_{u_F,-a_F} \cap P = F$. □

We are now ready to explore the combinatorial and geometric properties of the cones σ_Q. We begin with some combinatorial properties.

Proposition 3.4. *Let Q and Q' be faces of a lattice polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ of dimension n. Then:*

(a) *$Q \subseteq Q'$ if and only if $\sigma_{Q'} \subseteq \sigma_Q$.*

(b) *If $Q \subseteq Q'$, then $\sigma_{Q'}$ is a face of σ_Q, and all faces of σ_Q are of this form.*

(c) *$\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$, where Q'' is the smallest face of P containing Q and Q'.*

Proof. To prove part (a), note that if $Q \subseteq Q'$, then any facet containing Q' also contains Q, which implies $\sigma_{Q'} \subseteq \sigma_Q$. The other direction follows easily from Corollary 3.3 since every face is the intersection of the facets containing it by Proposition 2.1.

For part (b), fix a vertex $v \in Q$ and note that by (3.2), Q determines a face $\boldsymbol{Q}$ of C_v. Using the duality of Proposition 2.9 of Chapter 1, $\boldsymbol{Q}$ gives the dual face

$$\boldsymbol{Q}^* = C_v^\vee \cap \boldsymbol{Q}^\perp = \sigma_v \cap \boldsymbol{Q}^\perp$$

of the cone σ_v. Then using $\sigma_v = \mathrm{Cone}(u_F \mid v \in F)$ and $\boldsymbol{Q} \subseteq C_v = \sigma_v^\vee$, one obtains

$$\boldsymbol{Q}^* = \mathrm{Cone}(u_F \mid v \in F,\, \boldsymbol{Q} \subseteq H_{u_F,0}).$$

Since $v \in Q$, the inclusion $\boldsymbol{Q} \subseteq H_{u_F,0}$ is equivalent to $Q \subseteq H_{u_F,-a_F}$, which in turn is equivalent to $Q \subseteq F$. It follows that

$$\boldsymbol{Q}^* = \mathrm{Cone}(u_F \mid Q \subseteq F) = \sigma_Q, \tag{3.3}$$

so that σ_Q is a face of σ_v, and all faces of σ_v arise in this way.

In particular, $Q \subseteq Q'$ means that $\sigma_{Q'}$ is also a face of σ_v, and since $\sigma_{Q'} \subseteq \sigma_Q$ by part (a), we see that $\sigma_{Q'}$ a face of σ_Q. Furthermore, every face of σ_Q is a face of σ_v by Proposition 2.6 of Chapter 1 and hence is of the form $\sigma_{Q'}$ for some face Q'. Using part (a) again, we see that $Q \subseteq Q'$, and part (b) follows.

For part (c), let Q'' be the smallest face of P containing Q and Q'. This exists because a face is the intersection of the facets containing it, so that Q'' is the intersection of all facets containing Q and Q' (if there are no such facets, then $Q'' = P$). By part (b) $\sigma_{Q''}$ is a facet of both σ_Q and $\sigma_{Q'}$. Thus $\sigma_{Q''} \subseteq \sigma_Q \cap \sigma_{Q'}$.

It remains to prove the opposite inclusion. If $\sigma_Q \cap \sigma_{Q'} = \{0\} = \sigma_P$, then $Q'' = P$ and we are done. If $\sigma_Q \cap \sigma_{Q'} \neq \{0\}$, any nonzero u in the intersection lies in both σ_Q and $\sigma_{Q'}$. The proof of Proposition 3.6 given below will show that $H_{u,b}$ is a supporting affine hyperplane of P for some $b \in \mathbb{R}$. By Lemma 3.2, $u \in \sigma_Q$ and $u \in \sigma_{Q'}$ imply that Q and Q' lie in $H_{u,b} \cap P$. The latter is a face of P containing Q and Q', so that $Q'' \subseteq H_{u,b} \cap P$ since Q'' is the smallest such face. Applying Lemma 3.2 again, we see that $u \in \sigma_{Q''}$. □

Proposition 3.4 shows that there is a bijective correspondence between faces and cones. The proposition also implies that the cones fit together as follows.

Corollary 3.5. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a lattice polytope of dimension n and set $\Sigma_P = \{\sigma_Q \mid Q$ is a face of $Q\}$. Then:*

(a) *For all $\sigma_Q \in \Sigma_P$, each face of σ_Q is also in Σ_P.*

(b) *The intersection $\sigma_Q \cap \sigma_{Q'}$ of any two cones in Σ_P is a face of each.*

In Chapter 3, a collection of strongly convex rational polyhedral cones satisfying conditions (a) and (b) of Corollary 3.5 will be called a *fan*. Since the cones in the above fan Σ_P are built from the inward-pointing normal vectors u_F, we call Σ_P the *normal fan* or *inner normal fan* of P.

Before we draw pictures of normal fans, we mention some of the geometric properties of the cones σ_Q.

Proposition 3.6. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a lattice polytope of dimension n and consider the cones σ_Q in the normal fan Σ_P of P. Then:*

(a) $\dim Q + \dim \sigma_Q = n$ *for all faces Q of P.*

(b) $N_{\mathbb{R}} = \bigcup_v \sigma_v = \bigcup_{Q \text{ face of P}} \sigma_Q$.

Proof. Let Q be a face of P and take a vertex v of Q. By (3.2) this gives a face $\boldsymbol{Q}$ of the cone C_v, which has a dual face $\boldsymbol{Q}^*$ of the dual cone $C_v^\vee = \sigma_v$. Since $\boldsymbol{Q}^* = \sigma_Q$ by (3.3), we have

$$\dim Q + \dim \sigma_Q = \dim \boldsymbol{Q} + \dim \boldsymbol{Q}^* = n,$$

where the first equality uses Exercise 3.2 and the second follows from Proposition 2.9 of Chapter 1. This proves part (a). For part (b), let $u \in N_{\mathbb{R}}$ be nonzero and set $b = \min\{\langle v, u\rangle \mid v \text{ vertex of } P\}$. Then $P \subseteq H_{u,b}^+$ and $v \in H_{u,b}$ for at least one vertex of P, so that $u \in \sigma_v$ by Lemma 3.2. To prove the final equality of part (b), note that every face Q contains a vertex v, so that σ_Q is a face of σ_v by Proposition 3.4. □

In the terminology of Chapter 3, a fan satisfying the condition of part (b) of Proposition 3.6 is called *complete*. Thus the normal fan of a lattice polytope is always complete.

Example 3.7. Figure 6 on the next page shows a lattice hexagon P in the plane together with its normal fan. The vertices of P are labeled $v_1, \ldots, v_6$, with corresponding cone $\sigma_1, \ldots, \sigma_6$ in the normal fan. In the figure, P is shown on the right, and at each vertex v_i, we have drawn the normal vectors of the facets containing v_i and shaded the cone σ_i they generate. On the right, these cones are assembled at the origin to give the normal fan.

Notice how one can read off the structure of P from the normal fan. For example, two cones σ_i and σ_j share a ray in Σ_P if and only if the vertices v_i and v_j lie on an edge of P. ◇

Here is a similar example that illustrates an important property of normal fans.

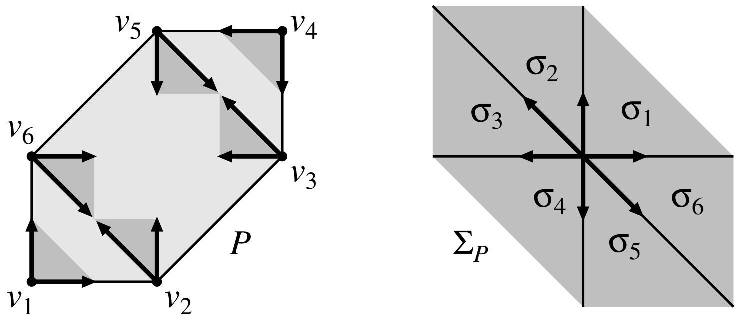


Figure 6. A lattice hexagon P and its normal fan Σ_P

Example 3.8. The 2-simplex $\Delta_2 \subseteq \mathbb{R}^2$ has vertices $0, e_1, e_2$. Let $P = k\Delta_2$ for some positive integer k. Figure 7 shows P and its normal fan Σ_P. At each vertex

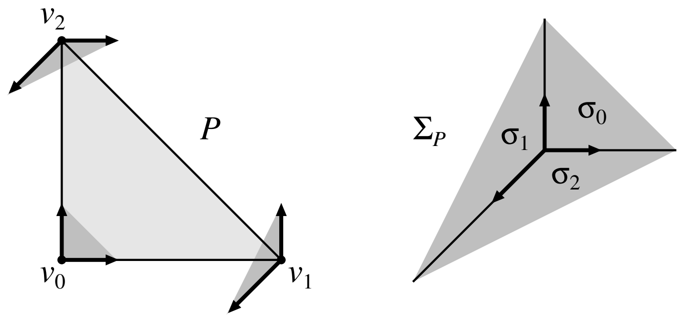


Figure 7. The triangle $P = k\Delta_2 \subseteq \mathbb{R}^2$ and its normal fan Σ_P

v_i of P, we have drawn the normal vectors of the facets containing v_i and shaded the cone σ_i they generate. The reassembled cones appear on the left as Σ_P. $\diamond$

An important observation is that the normal fan in Example 3.8 doesn't depend on the integer k. In general, multiplying a polytope by a positive integer has no effect on its normal fan, and the same is true for translations by lattice points. We record these properties in the following proposition (Exercise 3.3).

Proposition 3.9. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope. Then for any lattice point $m \in M$ and any integer $k \geq 1$, the polytopes $m + P$ and kP have the same normal fan as P.* $\square$

Here is an example of the normal fan of a polytope in $\mathbb{R}^3$.

Example 3.10. Consider the cube $P \subseteq \mathbb{R}^3$ with vertices $(\pm1, \pm1, \pm1)$. The facet normals are $\pm e_1, \pm e_2, \pm e_3$, and the facet presentation of P is

$$\langle m, \pm e_i \rangle \geq -1.$$

The origin is an interior point of P. By Exercise 2.1, the facet normals are the vertices of the dual polytope P°, the octahedron pictured in Figure 8.

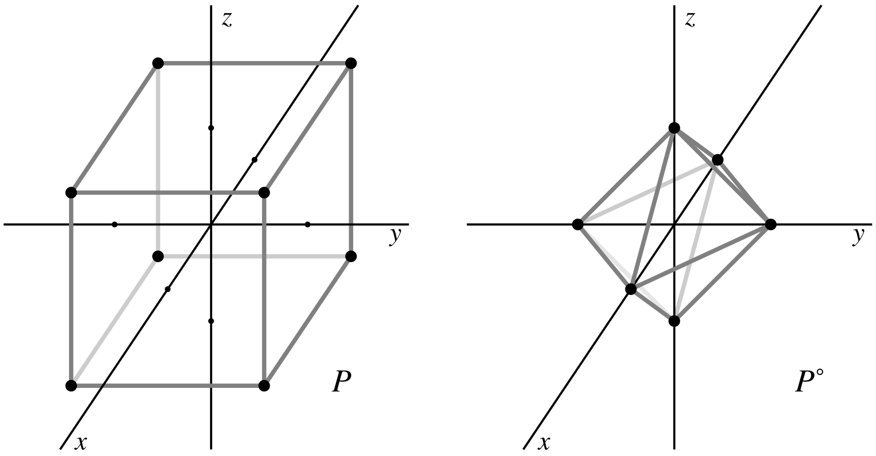


Figure 8. A cube $P \subseteq \mathbb{R}^3$ and its dual octahedron P°

However, the facet normals also give the normal fan of P, and one can check that in the above figure, the maximal cones of the normal fan are the octants of $\mathbb{R}^3$, which are just the cones over the facets of the dual polytope P°. ◇

As noted earlier, it is rare that both P and P° are lattice polytopes. However, whenever $P \subseteq M_\mathbb{R}$ is a lattice polytope containing 0 as an interior point, it is still true that maximal cones of the normal fan Σ_P are the cones over the facets of $P^\circ \subseteq N_\mathbb{R}$ (Exercise 3.4).

The special behavior of the polytopes P and P° discussed in Examples 2.6 and 3.10 leads to the following definition.

Definition 3.11. An n-dimensional lattice polytope $P \subseteq M_\mathbb{R} \simeq \mathbb{R}^n$ is ***reflexive*** if its facet presentation is

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F\rangle \geq -1 \text{ for all facets } F\}.$$

If P is reflexive, then 0 is a lattice point of P and and is in fact the *only* interior lattice point of P (Exercise 3.5). Since $a_F = 1$ for all F, Exercise 2.1 implies that

$$P^\circ = \mathrm{Conv}(u_F \mid F \text{ facet of } P).$$

Thus P° is a lattice polytope and is in fact reflexive (Exercise 3.5).

We will see later that reflexive polytopes lead to some very interesting toric varieties that are important for mirror symmetry.

The General Case. Earlier in the section, we defined the toric variety X_P of a very ample polytope and studied the affine pieces of X_P. This development leaves two items of unfinished business:

- What is the interesection two of the affine pieces described in Theorem 3.1?
- How do we define the projective toric variety X_P when P is not very ample?

As we will see, these questions are linked in a very satisfying manner.

To study the first question, let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional very ample polytope and set $s = |P \cap M|$. Then

$$X_P = X_{P\cap M} \subseteq \mathbb{P}^{s-1}.$$

Label the homogeneous coordinates of $\mathbb{P}^{s-1}$ as x_m for $m \in P \cap M$. If v is a vertex of P, then $U_v \subseteq \mathbb{P}^{s-1}$ is the affine open subset where $x_v \neq 0$, and Theorem 3.1 tells us that

$$X_P \cap U_v = U_{\sigma_v} = \mathrm{Spec}(\mathbb{C}[\sigma_v^\vee \cap M]).$$

Thus $X_P \cap U_v$ is the affine toric variety of the cone σ_v in the normal fan Σ_P of P.

Proposition 3.12. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be very ample of dimension n. If $v \neq w$ are vertices of X_P and Q is the smallest face of P containing v and w, then*

$$X_P \cap U_v \cap U_w = U_{\sigma_Q} = \mathrm{Spec}(\mathbb{C}[\sigma_Q^\vee \cap M])$$

and the inclusions

$$X_P \cap U_v \supseteq X_P \cap U_v \cap U_w \subseteq X_P \cap U_w$$

can be written

$$U_{\sigma_v} \supseteq (U_{\sigma_v})_{\chi^{w-v}} = U_{\sigma_Q} = (U_{\sigma_w})_{\chi^{v-w}} \subseteq U_{\sigma_w}. \tag{3.4}$$

Proof. We analyzed the intersection of affine pieces of $X_{P\cap M}$ in §1. Translated to the notation being used here, (1.6) and (1.7) imply that

$$X_P \cap U_v \cap U_w = (U_{\sigma_v})_{\chi^{w-v}} = (U_{\sigma_w})_{\chi^{v-w}}.$$

Thus all we need to show is that

$$(U_{\sigma_v})_{\chi^{w-v}} = U_{\sigma_Q}.$$

However, we have $w - v \in C_v = \sigma_v^\vee$, so that $\tau = H_{w-v} \cap \sigma_v$ is a face of σ_v. In this situation, Proposition 3.16 and equation (3.4) of Chapter 1 imply that

$$(U_{\sigma_v})_{\chi^{w-v}} = U_\tau.$$

Thus the proposition will follow once we prove $\tau = \sigma_Q$, i.e., $H_{w-v} \cap \sigma_v = \sigma_Q$. Since $\sigma_Q = \sigma_v \cap \sigma_w$ by Proposition 3.4, it suffices to prove that

$$H_{w-v} \cap \sigma_v = \sigma_v \cap \sigma_w.$$

Let $u \in H_{w-v} \cap \sigma_v$. If $u \neq 0$, there is $b \in \mathbb{R}$ such $H_{u,b}$ is a supporting affine hyperplane of P. Then $u \in \sigma_v$ implies $v \in H_{u,b}$ by Lemma 3.2, so that $w \in H_{u,b}$ since $u \in H_{w-v}$. Applying Lemma 3.2 again, we get $u \in \sigma_w$. Going the other way, let $u \in \sigma_v \cap \sigma_w$. If $u \neq 0$, pick $b \in \mathbb{R}$ as above. Then $u \in \sigma_v \cap \sigma_w$ and Lemma 3.2 imply that $v, w \in H_{u,b}$, from which $u \in H_{w-v}$ follows easily. This completes the proof. □

This proposition and Theorem 3.1 have the remarkable result that the normal fan Σ_P completely determines the internal structure of X_P: we build X_P from local pieces given by the affine toric varieties U_{σ_v}, glued together via (3.4). We don't need the ambient projective space $\mathbb{P}^{s-1}$ for any of this—everything we need to know is contained in the normal fan.

As a consequence, we can now give the general definition of the toric variety of a polytope.

Definition 3.13. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope. Then we define

$$X_P = X_{(kP)\cap M}$$

where k is any positive integer such that kP is very ample.

Such integers k exist by Corollary 2.17, and if k and ℓ are two such integers, then kP and ℓP have the same normal fan by Proposition 3.9, namely $\Sigma_{kP} = \Sigma_{\ell P} = \Sigma_P$. It follows that while $X_{(kP)\cap M}$ and $X_{(\ell P)\cap M}$ lie in different projective spaces, they are built from the affine toric varieties U_{σ_v} glued together via (3.4). Once we develop the language of abstract varieties in Chapter 3, we will see that X_P is well-defined as an abstract variety.

We will often speak of X_P without regard to the projective embedding. When we want to use a specific embedding, we will say "X_P is embedded using kP", where we assume that kP is very ample. In Chapters 5 and 6 we will use the language of divisors and line bundles to restate this in terms of a divisor D_P on X_P such that kD_P is very ample precisely when kP is.

Here is a simple example to illustrate the difference between X_P as an abstract variety and X_P as sitting is a specific projective space.

Example 3.14. Consider the n-simplex $\Delta_n \subseteq \mathbb{R}^n$. We can define X_{Δ_n} using any integer $k \geq 1$ since Δ_n is normal and hence very ample. The lattice points in $k\Delta_n$ correspond to the $s_k = \binom{n+k}{k}$ monomials of $\mathbb{C}[t_1, \dots, t_n]$ of total degree $\leq n$. This gives an embedding $X_{\Delta_n} \subseteq \mathbb{P}^{s_k-1}$. When $k = 1$, $\Delta_n \cap \mathbb{Z}^n = \{0, e_1, \dots, e_n\}$, which easily implies that

$$X_{\Delta_n} = \mathbb{P}^n.$$

The normal fan of Δ_n is described in Exercise 3.6. For an arbitrary $k \geq 1$, we can regard $X_{\Delta_n} \subseteq \mathbb{P}^{s_k-1}$ as the image of the map

$$\nu_k : \mathbb{P}^n \longrightarrow \mathbb{P}^{s_k-1}$$

defined using all monomials of total degree k in $\mathbb{C}[x_0,\ldots,x_n]$ (Exercise 3.6). It follows that this map is an embedding, usually called the *Veronese embedding*. But when we forget the embedding, the underlying toric variety is just $\mathbb{P}^n$.

The Veronese embedding allows us to construct some interesting affine open subsets of $\mathbb{P}^n$. Let $f \in \mathbb{C}[x_0,\ldots,x_n]$ be nonzero and homogeneous of degree k and write $f = \sum_{|\alpha|=k} c_\alpha x^\alpha$. We write the homogeneous coordinates of $\mathbb{P}^{s-1}$ as y_α for $|\alpha| = k$. Then $L = \sum_{|\alpha|=k} c_\alpha y_\alpha$ is a nonzero linear form in the variables y_α, so that $\mathbb{P}^{s_k-1} \setminus \mathbf{V}(L)$ is a copy of $\mathbb{C}^{s-1}$ (Exercise 3.6). If follows that

$$\mathbb{P}^n \setminus \mathbf{V}(f) \simeq \nu_k(\mathbb{P}^n) \cap \left(\mathbb{P}^{s_k-1} \setminus \mathbf{V}(L)\right)$$

is an affine variety (usually not toric). This shows that $\mathbb{P}^n$ has a richer supply of affine open subsets than just the open sets $U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i)$ considered earlier in the chapter. ◇

When we explain the Proj construction $\mathbb{P}^n$ later in the book, we will see the intrinsic reason why $\mathbb{P}^n \setminus \mathbf{V}(f)$ is an affine open subset of $\mathbb{P}^n$.

Example 3.15. The 2-dimensional analog of the rational normal curve C_n is the *rational normal scroll* $S_{a,b}$, which is the toric variety of the polygon

$$P_{a,b} = \mathrm{Conv}(0, ae_1, e_2, be_2 + e_2) \subseteq \mathbb{R}^2,$$

where $a, b \in \mathbb{N}$ satisfy $1 \leq a \leq b$. The polygon $P = P_{2,4}$ and its normal fan are pictured in Figure 9.

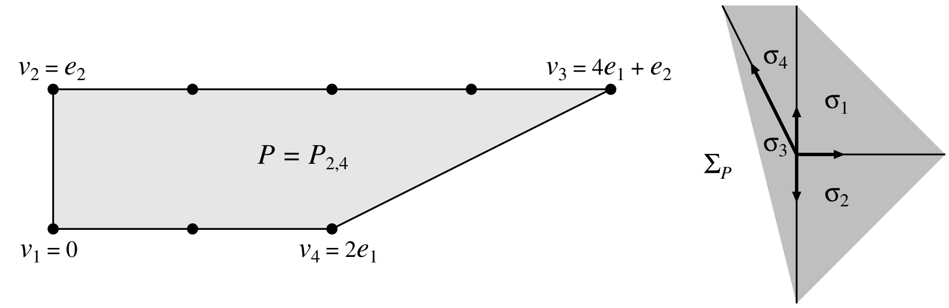


Figure 9. The polygon of a rational normal scroll and its normal fan

In general, the polygon $P_{a,b}$ has $a + b + 2$ lattice points and gives the map

$$(\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^{a+b+1}, \quad (s,t) \mapsto (1, s, s^2, \ldots, s^a, t, st, s^2t, \ldots, s^bt)$$

such that $S_{a,b} = X_{P_{a,b}}$ is the Zariski closure of the image. To describe the image, we rewrite the map as

$$\mathbb{C} \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{a+b+1}, \quad (s, \lambda, \mu) \mapsto (\lambda, s\lambda, s^2\lambda, \ldots, s^a\lambda, \mu, s\mu, s^2\mu, \ldots, s^b\mu).$$

When $(\lambda, \mu) = (1, 0)$, the map is $s \mapsto (1, s, s^2, \ldots, s^a, 0, \ldots, 0)$, which is the rational normal curve C_a mapped to the first $a+1$ coordinates of $\mathbb{P}^{a+b+1}$. In the same way, $(\lambda, \mu) = (0, 1)$ gives the rational normal curve C_b mapped to the last $b+1$ coordinates of $\mathbb{P}^{a+b+1}$. If we think of these two curves as the "edges" of a scroll, then fixing s gives a point on each edge, and letting $(\lambda, \mu) \in \mathbb{P}^1$ vary gives the line of the scroll connecting the two points. So it really is a scroll!

An important observation is that the normal fan depends *only* on the difference $b - a$, since this determines the slope of the slanted edge of $P_{a,b}$. If we denote the difference by $r \in \mathbb{N}$, it follows that as abstract toric varieties, we have

$$X_{P_{1,r+1}} = X_{P_{2,r+2}} = X_{P_{2,r+2}} = \cdots$$

since they are all constructed from the same normal fan. In Chapter 3, we will see that this is the Hirzebruch surface $\mathscr{H}_r$.

But if we think of the projective surface $S_{a,b} \subseteq \mathbb{P}^{a+b+1}$, then a and b have a unique meaning. For example, they have a strong influence on the defining equations of $S_{a,b}$. Let the homogeneous coordinates of $\mathbb{P}^{a+b+1}$ be $x_0, \ldots, x_a, y_0, \ldots, y_b$ and consider the $2 \times (a+b)$ matrix

$$\left(\begin{array}{cccc|cccc} x_0 & x_1 & \cdots & x_{a-1} & y_0 & y_1 & \cdots & y_{b-1} \\ x_1 & x_1 & \cdots & x_a & y_1 & y_1 & \cdots & y_b \end{array}\right).$$

One can show that $I(S_{a,b} \subseteq \mathbb{C}[x_0, \ldots, x_a, y_0, \ldots, y_b]$ is generated by the 2×2 minors of this matrix (see [**12**, Exercise 9.11], for example). ◇

Example 3.15 is another example of a determinantal variety, as is the rational normal curve from Example 0.1. Note that the rational normal curve C_n comes from the polytope $[0, n] = n\Delta_1$, where the underlying toric variety is just $\mathbb{P}^1$.

Exercises for §3.

3.1. This exercise will use the same notation as the proof of Theorem 3.1.

(a) Let $H_{u,a}$ be a supporting hyperplane of a vertex $m_i \in P$. Prove that $H_{u,0}$ is a supporting hyperplane of $0 \in C_i$

(b) Prove that $\dim C_i = \dim P$.

3.2. Consider the maps defined in (3.2).

(a) Show that these maps are inverses of each other and define a bijection between the faces of the cone C_v and the faces of P containing v.

(b) Prove that these maps preserve dimensions, inclusions, and intersections.

(c) Explain how this exercise relates to Exercise3.1.

3.3. Prove Proposition 3.9.

3.4. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope containing 0 as an interior point, and let $P^\circ \subset N_{\mathbb{R}}$ be its dual polytope. Prove that the normal fan Σ_P consists of the cones over the faces of P°. Hint: Exercise 2.1 will be useful.

3.5. Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.

(a) Prove that 0 is the only interior lattice point of P.

(b) Prove that $P^\circ \subset N_{\mathbb{R}}$ is reflexive.

3.6. This exercise is concerned with Example 3.14

(a) Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Prove that the normal fan of the standard n-simplex consists of the cones $\mathrm{Cone} S$ for all proper subsets $S \subseteq \{e_0, e_1, \dots, e_n\}$. Draw pictures of the normal fan for $n = 1, 2, 3$.

(b) For an integer $k \geq 1$, show that the variety $X_{k\Delta_n} \subseteq \mathbb{P}^{s_k - 1}$ is given by the map $\nu_k : \mathbb{P}^n \longrightarrow \mathbb{P}^{s_k-1}$defined using all monomials of total degree k in $\mathbb{C}[x_0, \dots, x_n]$.

3.7. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope and let $Q \subseteq P$ be a face. Prove the following intrinsic description of cone $\sigma_Q \in \Sigma_P$:

$$\sigma_Q - \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle \leq \langle m', u\rangle \text{ for all } m \in Q,\ m' in P\}.$$

3.8. Prove that all lattice rectangles in the plane with edges parallel to the coordinate axes have the same normal fan.

4. Properties of Projective Toric Varieties

We conclude this section by studying when the projective toric variety X_P of a polytope P is smooth or normal.

Normality. For the toric variety X_P of a polytope, we have the following basic result on normality.

Theorem 4.1. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope. Then:*

(a) *X_P is normal.*

(b) *X_P is projectively normal under the embedding given by kP if and only if kP is normal.*

Proof. Part (a) is immediate since X_P is the union of affine pieces U_{σ_v}, v a vertex of P, and U_{σ_v} is normal by Theorem 3.5 of Chapter 1. In Chapter 3 we will give an intrinsic definition of normality that will make this argument completely rigorous.

For part (b), the discussion following (1.4) shows that the projective embedding of X_P given by $X_{(kP)\cap M}$ has affine cone given by $Y_{((kP)\cap M)\times\{1\}}$. By Theorem 3.5 of Chapter 1, this is normal if and only if the semigroup $\mathbb{N}((kP) \cap M) \times \{1\}$ is saturated in $M \times \mathbb{Z}$, and since $((kP) \cap M) \times \{1\}$ generates the cone C_P, this is equivalent to saying that $(kP) \cap M) \times \{1\}$ generates the semigroup $C_P \cap (M \times \mathbb{Z})$. Then we are done by Lemma 2.12. $\square$

Smoothness. Given the results of Chapter 1, the smoothness of X_P is equally easy to determine. We need one definition.

Definition 4.2. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope.

(a) Given an a vertex v of P and an edge E containing v, let w_E be the first lattice point of E different from v encountered as one tranverses E starting at v. In other words, $w_E - v$ is the ray generator of the ray $\mathrm{Cone}(E - v)$.

(b) P is ***smooth*** if for every vertex v, the vectors $w_E - v$, where E is an edge of P containing v, form a subset of a basis of M. In particular, if $\dim P = \dim M_{\mathbb{R}}$, then the vectors $w_E - v$ form a basis of M.

We can now characterize when X_P is smooth.

Theorem 4.3. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope. Then the following are equivalent:*

(a) *X_P is a smooth projective variety.*

(b) *Σ_P is a smooth fan, meaning that every cone in Σ_P is smooth in the sense of Definition 2.14.*

(c) *P is a smooth polytope.*

Proof. Smoothness is a local condition, so that a variety is smooth if and only if its local pieces are smooth. Thus X_P is smooth if and only if U_{σ_v} is smooth for every vertex v of P, and U_{σ_v} is smooth if and only if σ_v is smooth. Since faces of smooth cones are smooth and Σ_P consists of the σ_v and their faces, the equivalence (a) $\Leftrightarrow$ (b) follows immediately.

For (b) $\Leftrightarrow$ (c), first observe that σ_v is smooth if and only if it dual $C_v = \sigma_v^\vee$ is smooth. The discussion following (3.2) makes it easy to see that the ray generators of C_v are the vectors $w_E - v$ from Definition 4.2. It follows immediately that P is smooth if and only if C_v is smooth for every vertex v, and we are done. □

The theorem makes it easy to check the smoothness of simple examples such as the toric variety of the hexagon in Example 3.7 or the rational normal scroll $S_{a,b}$ of Example 3.15 (Exercise 4.1).

We also note the following useful fact, which you will prove in Exercise 4.2

Proposition 4.4. *Every smooth n-dimensional lattice polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ is very ample.* □

Here is an example of a smooth reflexive polytope whose dual is not smooth.

Example 4.5. Let $P = (n+1)\Delta_n - (1, \dots, 1) \subseteq \mathbb{R}^n$, where Δ_n is the standard n simplex. Proposition 3.9 implies that P and Δ_n have the same normal fan, so that P and X_P are smooth.

The polytope P has the following further interesting properties (Exercise 4.3). First, P has the facet presentation

$$\begin{aligned} x_i &\geq -1, \quad i = 1, \ldots, n, \\ -x_1 - \cdots - x_n &\geq -1, \end{aligned}$$

so that P is reflexive with dual

$$P^\circ = \mathrm{Conv}(e_0, e_1, \ldots, e_n), \quad e_0 = -e_1 - \cdots - e_n.$$

The normal fan of P° consists of cones over the faces of P. In particular, the cone of Σ_{P° corresponding to the vertex $e_0 \in P^\circ$ is the cone

$$\sigma_{e_0} = \mathrm{Conv}(v_1, \ldots, v_n), \quad v_i = e_0 + (n+1)e_i.$$

Figure 10 shows P and the cone σ_{e_0} when $n = 2$.

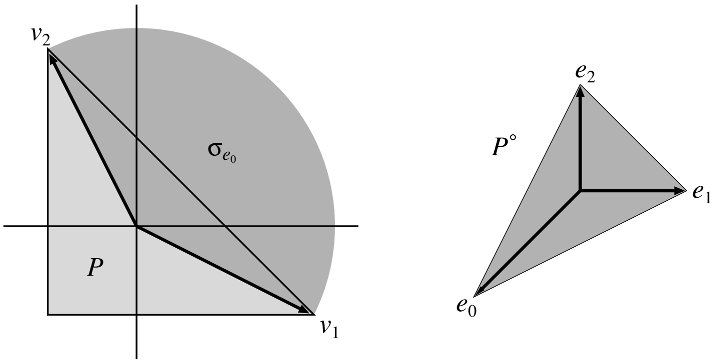


Figure 10. The cone σ_{e_0} of the normal fan of P°

For general n, observe that $v_i - v_j = (n+1)(e_i - e_j)$. This makes it easy to see that $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ has index $(n+1)^{n-1}$ in $\mathbb{Z}^n$. Thus σ_{e_0} is not smooth when $n \geq 2$. It follows that the "dual" toric variety X_{P° is singular for $n \geq 2$. Later we will construct X_{P° as the quotient of $\mathbb{P}^n$ under the action of a finite group $G \simeq (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$. ◇

Example 4.6. Consider $P = \mathrm{Conv}(0, 2e_1, e_2) \subseteq \mathbb{R}^2$. Since P is very ample, the lattice points $P \cap \mathbb{Z}^2 = \{0, e_1, 2e_1, e_2\}$ give the map $(\mathbb{C}^*)^2 \to \mathbb{P}^3$ defined by

$$(s,t) \longmapsto (1, s, s^2, t)$$

such that X_P is the Zariski closure of the image. If $\mathbb{P}^3$ has homogeneous coordinates y_0, y_1, y_2, y_3, then we have

$$X_P = \mathbf{V}(y_0 y_2 - y_1^2) \subseteq \mathbb{P}^3.$$

Comparing this to Example 0.5, we see that X_P is the weighted projective space $\mathbb{P}(1,1,2)$. Later we will learn the systematic reason why this is true.

The variety X_P is not smooth. By working on the affine piece $X_P \cap U_3$, one can check directly that $(0,0,0,1)$ is a singular point of X_P.

We can also use Theorem 4.3 and the normal fan of P, shown in Figure 11. One can check that the cones σ_0 and σ_1 are smooth, but σ_2 is not, so that Σ_P is

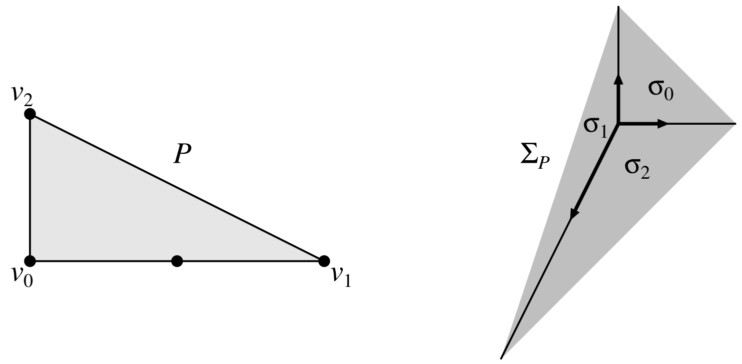


Figure 11. The polygon giving $\mathbb{P}(1,1,2)$ and its normal fan

not a smooth fan. In terms of P, note that the vectors from v_2 to the first lattice points along the edges containing v_2 do not generate $\mathbb{Z}^2$. Either way, Theorem 4.3 implies that X_P is not smooth.

If you look carefully, you will see that σ_2 is the *only* nonsmooth cone of the normal fan Σ_P. Once we study the correspondence between cones and orbits in Chapter 3, we will see that the non-smooth cone σ_2 corresponds to the singular point $(0,0,0,1)$ of X_P. $\diamond$

Products of Projective Toric Varieties. Our final task is to understand the toric variety of a product of polytopes. Let $P_i \subseteq (M_i)_{\mathbb{R}} \simeq \mathbb{R}^{n_i}$ be lattice polytopes with $\dim P_i = n_i$ for $i = 1, 2$. This gives a lattice polytope $P_1 \times P_2 \subset (M_1 \times M_2)_{\mathbb{R}}$ of dimension $n_1 + n_2$.

Replacing P_1 and P_2 with suitable multiples, we can assume that P_1 and P_2 are very ample. This gives projective embeddings

$$X_{P_i} \hookrightarrow \mathbb{P}^{s_i - 1}, \quad s_i = |P_i \cap M_i|,$$

so that by Proposition 0.4, $X_{P_1} \times X_{P_2}$ is a subvariety of $\mathbb{P}^{s_1-1} \times \mathbb{P}^{s_1-1}$. Using the Segre embedding

$$\mathbb{P}^{s_1-1} \times \mathbb{P}^{s_2-1} \hookrightarrow \mathbb{P}^{s-1}, \quad s = s_1 s_2,$$

we get an embedding

$$X_{P_1} \times X_{P_2} \hookrightarrow \mathbb{P}^{s-1}. \tag{4.1}$$

We can understand this projective variety as follows.

Theorem 4.7. *If P_1 and P_2 are very ample, then*

(a) $P_1 \times P_2 \subset (M_1 \times M_2)_{\mathbb{R}}$ *is a very ample polytope with lattice points*

$$(P_1 \times P_2) \cap (M_1 \times M_2) = (P_1 \cap M_1) \times (P_2 \cap M_2).$$

Thus the integer s defined above is $s = |(P_1 \times P_2) \cap (M_1 \times M_2)|$.

(b) *The image of the embedding $X_{P_1} \times X_{P_2} \hookrightarrow \mathbb{P}^{s-1}$ coming from the very ample polytope $P_1 \times P_2$ equals the image of* (4.1).

(c) $X_{P_1 \times P_2} \simeq X_{P_1} \times X_{P_2}$.

Proof. For part (a), the assertions about lattice points are clear. The vertices of $P_1 \times P_2$ consist of ordered pairs (v_1, v_2) where v_i is a vertex of P_i (Exercise 4.4). Given such a vertex, we have

$$(P_1 \times P_2) \cap (M_1 \times M_2) - (v_1, v_2) = (P_1 \cap M_1 - v_1) \times (P_2 \cap M_2 - v_2).$$

Since P_i is very ample, we know that $\mathbb{N}(P_i \cap M_i - v_i)$ is saturated in M_i. From here, it follows easily that $P_1 \times P_2$ is very ample.

For part (b), let T_{N_i} be the torus of X_{P_i}. Since T_{N_i} is Zariski dense in X_{P_i}, it follows that $T_{N_1} \times T_{N_2}$ is Zariski dense in $X_{P_1} \times X_{P_2}$ (Exercise 4.4). When combined with the Segre embedding, it follows that $X_{P_1} \times X_{P_2}$ is the Zariski closure of the image of the map

$$T_{N_1} \times T_{N_2} \longrightarrow \mathbb{P}^{s_1 s_2 - 1}$$

given by the characters $\chi^m \chi^{m'}$, where m ranges over the s_1 elements of $P_1 \cap M_1$ and m ranges over the s_2 elements of $P_2 \cap M_2$. When we identify $T_{N_1} \times T_{N_2}$ with $T_{N_1 \times N_2}$, the product $\chi^m \chi^{m'}$ becomes the character $\chi^{(m,m')}$, so that the above map coincides with the map

$$T_{N_\times N_2} \longrightarrow \mathbb{P}^{s-1}$$

coming from the product polytope $P_1 \times P_2 \subset (M_1 \times M_2)_{\mathbb{R}}$. Part (b) follows, and part (c) is an immediate consequence. $\square$

Here is an obvious example.

Example 4.8. Since $\mathbb{P}^n$ is the toric variety of the standard n-simplex Δ_n, it follows that $\mathbb{P}^n \times \mathbb{P}^n$ is the toric variety of $\Delta_n \times \Delta_m$.

This also works for more than two factors. Thus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the toric variety of the cube pictured in Figure 8.

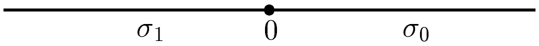
To have a complete theory of products, we need to know what happens to the normal fan. Here is the result, whose proof is left to the reader (Exercise 4.5).

Proposition 4.9. *Let $P_i \subseteq (M_i)_{\mathbb{R}} \simeq \mathbb{R}^{n_i}$ be lattice polytopes with $\dim P_i = n_i$ for $i = 1, 2$. Then*

$$\Sigma_{P_1 \times P_2} = \Sigma_{P_1} \times \Sigma_{P_2}. \qquad \square$$

Here is an easy example.

Example 4.10. The normal fan of an interval $[a, b] \subset \mathbb{R}$, where $a < b$ in $\mathbb{Z}$, is given by



The corresponding toric variety is $\mathbb{P}^1$. The cartesian product of two such intervals is a lattice rectangle whose toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$ by Theorem 4.7. If we set $\sigma_{ij} = \sigma_i \times \sigma_j$, then Proposition 4.9 gives the normal fan given in Figure 12.

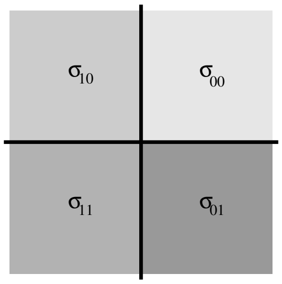


Figure 12. The normal fan of a lattice rectangle giving $\mathbb{P}^1 \times \mathbb{P}^1$

We will revisit this example in Chapter 3 when we construct toric varieties directly from fans. $\Diamond$

Proposition 4.9 suggests a different way to think about the product. Let v_i range over the vertices of P_i. Then the σ_{v_i} are the maximal cones in the normal fan Σ_{P_i}, which implies that

$$X_{P_i} = \textstyle\bigcup_{v_i} U_{\sigma_{v_i}}. \tag{4.2}$$

Thus

$$\begin{aligned} X_{P_1} \times X_{P_2} &= \big(\textstyle\bigcup_{v_1} U_{\sigma_{v_1}}\big) \times \big(\textstyle\bigcup_{v_2} U_{\sigma_{v_2}}\big) \\ &= \textstyle\bigcup_{(v_1,v_2)} U_{\sigma_{v_1}} \times U_{\sigma_{v_2}} \\ &= \textstyle\bigcup_{(v_1,v_2)} U_{\sigma_{v_1} \times \sigma_{v_2}} \\ &= \textstyle\bigcup_{(v_1,v_2)} U_{\sigma_{(v_1,v_2)}} = X_{P_1 \times P_2}. \end{aligned}$$

In this sequence of equalities, the first follows from (4.2), the second is obvious, the third uses Exercise 3.13 of Chapter 1, the fourth uses Proposition 4.9, and the last follows since (v_1, v_2) ranges over all vertices of $P_1 \times P_2$.

This argument shows that we can construct cartesian products of varieties using affine open covers, which reduces to the cartesian product of affine varieties defined in Chapter 1. We will use this idea in Chapter 3 to define the cartesian product of abstract varieties.

***Exercises for* §4.**

4.1. Show that the hexagon $P = \mathrm{Conv}(0, e_1, e_2, 2e_1 + e_2, e_1 + 2e_2, 2e_1 + 2e_2)$ pictured in Figure 6 and the trapezoid $P_{a,b}$ pictured in Figure 9 are smooth polygons. Also, of the polytopes shown in Figure 8, determine which ones are smooth.

4.2. Prove Proposition 4.4.

4.3. Consider the polytope $P = (n + 1)\Delta_n - (1, \ldots, 1)$ from Example 4.5.

(a) Verify the facet presentation of P given in the example.

(b) What is the facet presentation of P°? Hint: You know the vertices of P.

(c) Let $v_i = e_0 + (n + 1)e_i$, where $i = 1, \ldots, n$ and $e_0 = -e_1 - \cdots - e_n$, and let $L = \mathbb{Z}v_1+\cdots+\mathbb{Z}v_n$. Use the hint given in the text to prove $\mathbb{Z}^n/L \simeq (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$. This shows that the index of L in $\mathbb{Z}^n$ is $(n + 1)^{n_1}$, as claimed in the text.

4.4. Let $P_i \subseteq (M_i)_\mathbb{R} \simeq \mathbb{R}^{n_i}$ be lattice polytopes with $\dim P_i = n_i$ for $i = 1, 2$. Also let S_i be the set of vertices of P_i.

(a) Use supporting hyperplanes to prove that every element of $S_1 \times S_2$ is a vertex of $P_1 \times P_2$.

(b) Prove that $P_1 \times P_2 = \mathrm{Conv}(S_1 \times S_2)$ and conclude that $S_1 \times S_2$ is the set of vertices of $P_1 \times P_2$.

4.5. The goal of this exercise is to prove Proposition 4.9. We know from Exercise 4.4 that the vertices of $P_1 \times P_2$ are the ordered pairs (v_1, v_2) where v_i is a vertex of P_i.

(a) Adapt the argument of part (a) of Theorem 4.7 to show that $C_{(v_1,v_2)} = C_{v_1} \times C_{v_2}$. Taking duals, we see that the maximal cones of $\Sigma_{P_1\times P_2}$ are $\sigma_{(v_1,v_2)} = \sigma_{v_1} \times \sigma_{v_2}$.

(b) Given rational polyhedral cones $\sigma_i \subseteq (N_i)_\mathbb{R}$ and faces $\tau_i \subseteq \sigma_i$, prove that $\tau_1 \times \tau_2$ is a face of $\sigma_1 \times \sigma_2$ and that all faces of $\sigma_1 \times \sigma_2$ arise this way.

(c) Prove that $\Sigma_{P_1\times P_2} = \Sigma_{P_1} \times \Sigma_{P_2}$.

4.6. Consider positive integers $1 = q_0 \leq q_2 \leq \cdots \leq q_n$ with the property that $q_i | \sum_{i=0}^n q_i$ for $i = 0, \ldots, n$. Set $k_i = \big(\sum_{i=0}^n q_i\big)/q_i$ for $i = 1, \ldots, n$ and let

$$P_{q_0,\ldots,q_n} = \mathrm{Conv}(0, k_1e_1, k_2e_2, \ldots, k_ne_n) - (1, \ldots, 1).$$

Prove that $P_{q_0,\ldots,q_n}$ is reflexive and explain how it relates to Example 4.5. We will prove later that the toric variety of this polytope is the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$.

4.7. The *Sylvester sequence* is defined by $a_0 = 2$ and $a_{k+1} = 1 + a_1a_2\cdots a_k$. It begins $2, 3, 7, 43, 1807, \ldots$ and is described in [**24**, A000058]. Now fix a positive integer $n \geq 3$ and define $q_0, \ldots, q_n$ by $q_0 = q_1 = 1$ and $q_i = 2(a_{n-1} - 1)/a_{n-i}$ for $i = 2, \ldots, n$. For $n = 3$ and 4 this gives $1, 1, 4, 6$ and $1, 1, 12, 28, 42$. Prove that $q_0, \ldots, q_n$ satisfies the conditions of Exercise 4.6 and hence gives a reflexive simplex, denoted $S_{Q'_n}$ in [**19**]. This paper proves when $n \geq 4$, $S_{Q'_n}$ has the largest volume of all n-dimensional reflexive simplices and conjectures that it also has the largest number of lattice points.

Chapter 8

Toric Surfaces

Chapter 9

Resolution of Singularities

Chapter 10

Geometric Invariant Theory

Chapter 11

The Riemann-Roch Theorem

Chapter 12

The Topology of Toric Varieties

Chapter 13

Multigraded Commutative Algebra

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