Lecture 3
Projective Varieties and Abstract Varieties

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This lecture will review projective varieties in $\mathbb{P}^n$ and discuss how abstract varieties are constructed by gluing affine varieties together. We conclude with some observations about gluing affine toric varieties.

§1. Projective Varieties

Projective Space. We define *n-dimensional projective space* to be the set

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $\sim$ is the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ given by

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff \text{there is } \lambda \in \mathbb{C}^* \text{ with } (a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n). \tag{1.1}$$

Here, we use $\mathbb{C}^*$ to denote $\mathbb{C} - \{0\}$, which is a group under multiplication. As we vary $\lambda \in \mathbb{C}^*$, the points $\lambda(b_0, \ldots, b_n)$ lie on a line through the origin. Thus we get a bijection

$$\mathbb{P}^n \simeq \{\text{lines through the origin in } \mathbb{C}^{n+1}\}.$$

Exercise 1.1. $\mathbb{P}^n$ contains the subset $(\mathbb{C}^*)^{n+1}/\sim$. Note also that $(\mathbb{C}^*)^{n+1}$ is a group under component-wise multiplication.

a. Show that on $(\mathbb{C}^*)^{n+1}$, the equivalence classes of $\sim$ are the cosets of the subgroup $H = \{(\lambda, \ldots, \lambda) \mid \lambda \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^{n+1}$. Conclude that $(\mathbb{C}^*)^{n+1}/H \subset \mathbb{P}^n$.
b. Construct a group isomorphism $(\mathbb{C}^*)^{n+1}/H \simeq (\mathbb{C}^*)^n$.

Exercise 1.1 shows that $\mathbb{P}^n$ contains an isomorphic copy of $(\mathbb{C}^*)^n$. In Lecture 2, we saw that every n-dimensional affine toric variety contains the torus $(\mathbb{C}^*)^n$. We will see in Lecture 4 that $\mathbb{P}^n$ is a classic example of a non-affine toric variety.

We note that $\mathbb{P}^n$ has a *classical topology* inherited from the usual topology on $\mathbb{C}^{n+1} - \{0\}$.

Exercise 1.2. Let S^{2n+1} be the unit $(2n+1)$-sphere centered at the origin in $\mathbb{C}^{n+1}$.

a. Show that the natural map $S^{2n+1} \to \mathbb{P}^n$ is onto and conclude that $\mathbb{P}^n$ is compact.
b. Show that the fibers of $S^{2n+1} \to \mathbb{P}^n$ are isomorphic to S^1. This is the *Hopf fibration*.

Homogeneous Coordinates. A point p of $\mathbb{P}^n$ will be written $(a_0, \ldots, a_n)$. This is only unique up to the equivalence relation (1.1). We call $(a_0, \ldots, a_n)$ *homogeneous coordinates* of p. In some books, this is written $p = [a_0, \ldots, a_n]$ or $p = (a_0 : \ldots : a_n)$ to emphasize the non-unique nature of these coordinates. We prefer to write $p = (a_0, \ldots, a_n)$, where it will be clear from the context that we are using homogeneous coordinates.

Projective Varieties and Homogeneous Ideals. A polynomial $f \in \mathbb{C}[x_0,\ldots,x_n]$ is *homogeneous of degree* d if every term of f has total degree d. This is equivalent to the identity

$$f(\lambda x_0,\ldots,\lambda x_n) = \lambda^d f(x_0,\ldots,x_n). \tag{1.2}$$

Exercise 1.3. Show that any $f \in \mathbb{C}[x_0,\ldots,x_n]$ can be written uniquely in the form $f = \sum_{d\geq 0} f_d$ where f_d is homogeneous of degree d. We call f_d the *homogeneous components* of f.

Now suppose that $f \in \mathbb{C}[x_0,\ldots,x_n]$ is homogeneous of degree d. Given $p \in \mathbb{P}^n$, we can't define "$f(p)$" since using $p = (a_0,\ldots,a_n)$ would give

$$f(p) = f(a_0,\ldots,a_n),$$

while using $p = \lambda(a_0,\ldots,a_n)$ would give

$$f(p) = f(\lambda a_0,\ldots,\lambda a_n) = \lambda^d f(a_0,\ldots,a_n).$$

However, the equation $f(p) = 0$ is well-defined since $\lambda \in \mathbb{C}^*$. Thus, homogeneous polynomials $f_1,\ldots,f_s \in \mathbb{C}[x_0,\ldots,x_n]$ define the *projective variety*

$$\mathbf{V}(f_1,\ldots,f_s) = \{a \in \mathbb{P}^n \mid f_1(a) = \cdots = f_s(a) = 0\} \subset \mathbb{P}^n.$$

To formulate this in terms of ideals, we say that an ideal $I \subset \mathbb{C}[x_0,\ldots,x_n]$ is *homogeneous* if it is generated by homogeneous polynomials.

Exercise 1.4. Show that an ideal $I \subset \mathbb{C}[x_0,\ldots,x_n]$ is homogeneous if and only if for all $f \in \mathbb{C}[x_0,\ldots,x_n]$, we have $f \in I \Leftrightarrow I$ contains the homogeneous components of f.

If $I \subset \mathbb{C}[x_0,\ldots,x_n]$ is a homogeneous ideal, then we have the projective variety

$$\mathbf{V}(I) = \{a \in \mathbb{C}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

Conversely, given an projective variety $V \subset \mathbb{C}^n$, we get the homogeneous ideal

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_0,\ldots,x_n] \mid f(a) = 0 \text{ for all } a \in V\}.$$

Exercise 1.5. We call $\langle x_0,\ldots,x_n\rangle \subset \mathbb{C}[x_0,\ldots,x_n]$ the *irrelevant ideal*. Show that $\mathbf{V}(I) = \emptyset$ whenever I contains a power of the irrelevant ideal.

Exercise 1.5 is actually part of the projective version of the Nullstellensatz, which goes as follows. We refer the reader to [1, Chapter 8] for a proof.

Projective Nullstellensatz. *Let $I \subset \mathbb{C}[x_0,\ldots,x_n]$ be a homogeneous ideal.*

a. $\mathbf{V}(I) = \emptyset$ *if and only if* $\langle x_0,\ldots,x_n\rangle^m \subset I$ *for some* $m \geq 0$.
b. $\mathbf{V}(I) \neq \emptyset$ *implies* $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$.

Most of the concepts defined for affine varieties in $\mathbb{C}^n$ can be extended to projective varieties in $\mathbb{P}^n$ in the obvious way:

- $W \subset V$ is a *subvariety* of a projective variety $V \subset \mathbb{P}^n$ if W is a projective variety in $\mathbb{P}^n$.
- If $V \subset \mathbb{P}^n$ is a projective variety, then we call $\mathbb{P}^n - V$ a *Zariski open* subset of $\mathbb{P}^n$.
- The *Zariski topology* is the topology on $\mathbb{P}^n$ whose open sets are the Zariski open sets.
- The *Zariski closure* $\overline{S}$ of a subset $S \subset \mathbb{P}^n$ is the smallest projective variety containing S.

Rational Functions on Projective Space. We've already seen that a homogeneous polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ does not give a function on $\mathbb{P}^n$. However, the quotient of two such polynomials works, provided they have the same degree. More precisely, suppose that $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ have degree d and that $g \neq 0$. Then (1.2) shows that we get a well-defined function

$$\frac{f}{g} : \mathbb{P}^n - \mathbf{V}(g) \longrightarrow \mathbb{C}$$

As in Lecture 1, we write this as $f/g : \mathbb{P}^n \dashrightarrow \mathbb{C}$ and say that f/g is a *rational function* on $\mathbb{P}^n$.

Exercise 1.6. The set of all rational functions on $\mathbb{P}^n$ is

$$\mathbb{C}(\mathbb{P}^n) = \Big\{\frac{f}{g} \in \mathbb{C}(x_0, \ldots, x_n) \mid f, g \text{ are homogeneous of degree } d \text{ and } g \neq 0\Big\}.$$

Prove that $\mathbb{C}(\mathbb{P}^n)$ is subfield of $\mathbb{C}(x_0, \ldots, x_n)$.

Mappings Between Projective Varieties. Suppose that $V \subset \mathbb{P}^n$ is a projective variety and $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials all of the same degree. Then we say that $f_0, \ldots, f_m$ have no *base points* on V if $V \cap \mathbf{V}(f_0, \ldots, f_m) = \emptyset$.

Exercise 1.7. Suppose that $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous of degree d and have no base points on V. Prove that the $(a_0, \ldots, a_n) \mapsto (f_0(a_0, \ldots, a_n), \ldots, f_m(a_0, \ldots, a_n))$ induces a well-defined function $F : V \longrightarrow \mathbb{P}^m$.

An important fact is that in the situation of Exercise 1.6, the image $F(V) \subset \mathbb{P}^m$ is a projective subvariety. When $V = \mathbb{P}^n$, this is proved in [1, Chapter 8], and the proof extends easily to cover the general case.

Exercise 1.8. When $V \subset \mathbb{C}^n$ and $F : V \to \mathbb{C}^m$ is a polynomial map, the image $F(V) \subset \mathbb{C}^m$ need not be a subvariety. For example, suppose that $V = \mathbf{V}(xy - 1) \subset \mathbb{C}^2$ and $F : V \to \mathbb{C}$ is $F(x, y) = x$. Prove that $F(V)$ is not a subvariety of $\mathbb{C}$. The fact that $F(V)$ is a subvariety in the projective case is one reason why projective varieties are so useful in algebraic geometry.

Affine Open Subsets. We can regard $\mathbb{P}^n$ as a union of affine spaces as follows. For $0 \leq i \leq n$, consider the Zariski open set $U_i = \mathbb{P}^n - \mathbf{V}(x_i)$.

Exercise 1.9. As above, $U_i = \mathbb{P}^n - \mathbf{V}(x_i) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid a_i \neq 0\}$.
a. Show that $U_i \simeq \mathbb{C}^n$ via $(a_0, \ldots, a_n) \mapsto (a_0/a_i, \ldots, a_{i-1}/a_i, a_{i-1}/a_i, \ldots, a_n/a_i)$.
b. Show that $\mathbf{V}(x_i) \simeq \mathbb{P}^{n-1}$ via $(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.
c. Show that $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$.

This exercise shows that we can regard $\mathbb{P}^n$ as $\mathbb{C}^n$ together with a copy of $\mathbb{P}^{n-1}$ "at infinity". Also, the open cover of Exercise 1.9 shows that projective varieties are unions of affine varieties.

Exercise 1.10. Let $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{P}^n$ be a projective variety. Prove that under the map $U_i \simeq \mathbb{C}^n$ from Exercise 1.9, $V \cap U_i$ corresponds to an affine variety defined by the vanishing of $\tilde{f}_j$, where $\tilde{f}_j(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f_j(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$. We call $\tilde{f}_j$ the *dehomogenization* of f_j with respect to x_i.

Another way to think about $U_i \simeq \mathbb{C}^n$ is to use $x_0/x_i, \ldots, x_{i-1}/x_i, x_{i-1}/x_i, \ldots, x_n/x_i$ as variables on $\mathbb{C}^n$. Then the dehomogenization map of Exercise 1.10 is just $f \mapsto f/x_i^d$, where $f \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous of degree d. This approach preserves rational functions.

Exercise 1.11. Show that the map $f/g \mapsto (f/x_i^d)/(g/x_i^d)$ induces an isomorphism of fields
$$\mathbb{C}(\mathbb{P}^n) \simeq \mathbb{C}(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i-1}/x_i, \ldots, x_n/x_i),$$
where $\mathbb{C}(\mathbb{P}^n)$ is the field of rational functions defined in Exercise 1.6.

Weighted Projective Space. We next discuss a generalization of $\mathbb{P}^n$. Given positive integers $q_0, \ldots, q_n$ satisfying $\gcd(q_0, \ldots, q_n) = 1$, we get the *weighted projective space*
$$\mathbb{P}(q_0, \ldots, q_n) = (\mathbb{C}^{n+1} - \{0\})/\sim,$$
where $\sim$ is the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ given by
$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff \text{there is } \lambda \in \mathbb{C}^* \text{ with } (a_0, \ldots, a_n) = (\lambda^{q_0} b_0, \ldots, \lambda^{q_n} b_n).$$
Obviously $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$. We will eventually show that $\mathbb{P}(q_0, \ldots, q_n)$ is a toric variety. The following exercise shows that $\mathbb{P}(q_0, \ldots, q_n)$ contains a copy of $(\mathbb{C}^*)^n$.

Exercise 1.12. As above, let $q_0, \ldots, q_n$ be positive integers with $\gcd(q_0, \ldots, q_n) = 1$.
a. Prove that $(\mathbb{C}^*)^{n+1}/\widetilde{H} \subset \mathbb{P}(q_0, \ldots, q_n)$, where $\widetilde{H} = \{(\lambda^{q_0}, \ldots, \lambda^{q_n}) \mid \lambda \in \mathbb{C}^*\}$.
b. Prove that $(\mathbb{C}^*)^{n+1}/\widetilde{H} \simeq (\mathbb{C}^*)^n$. Hint: Make $(q_0, \ldots, q_n)$ the first column of a matrix $M \in \mathrm{GL}(n+1, \mathbb{Z})$ and use M to define an automorphism of $(\mathbb{C}^*)^{n+1}$.

We call $q_0, \ldots, q_n$ the *weights* of the weighted projective space. In terms of the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, this means that x_i has degree q_i, and $f \in \mathbb{C}[x_0, \ldots, x_n]$ is *weighted homogeneous* of (weighted) degree d if
$$f(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \ldots, x_n). \tag{1.2}$$
It is then easy to see that we can define *weighted projective subvarieties* in $\mathbb{P}(q_0, \ldots, q_n)$ using weighted homogeneous polynomials.

There are several ways to think about weighted projective spaces. The following two exercises give two ways to represent $\mathbb{P}(1,1,2)$.

Exercise 1.13. Consider $\mathbb{P}(1,1,2)$ with variables x_0, x_1, x_2 of degrees $1,1,2$ repsectively.
a. Show that $x_0^2, x_0x_1, x_1^2, x_2$ are (weighted) homogeneous of degree 2.
b. Show that $(a_0, a_1, a_2) \mapsto (a_0^2, a_0a_1, a_1^2, a_2)$ is a well-defined map $F : \mathbb{P}(1,1,2) \to \mathbb{P}^3$.
c. Show that the map F of part b is injective and that its image is the surface defined by the equation $y_0y_2 - y_1^2 = 0$ (where y_0, y_1, y_2, y_3 are the coordinates of $\mathbb{P}^3$).

Exercise 1.14. Show that $(a_0, b_0, c_0) \to (a_0, b_0, c_0^2)$ gives a well-defined map $\mathbb{P}^2 \to \mathbb{P}(1,1,2)$. Also show that this map is surjective and is two-to-one except above $(0,0,1) \in \mathbb{P}(1,1,2)$.

We can also cover a weighted projective space by affine open subsets, though in this case, the open sets will be affine toric varieties instead of affine space $\mathbb{C}^n$. Rather than work this out in general, we will restrict to the case of $\mathbb{P}(1,1,2)$. Here, we have the Zariski open sets $U_i = \{(a_0, a_1, a_2) \in \mathbb{P}(1,1,2) \mid a_i \neq 0\}$.

Exercise 1.15. Let U_0, U_1, U_2 be the subsets of $\mathbb{P}(1,1,2)$ defined above.
a. Show that $U_0 \simeq \mathbb{C}^2$ via $(a,b,c) \mapsto (b/a, c/a^2)$ and $U_1 \simeq \mathbb{C}^2$ via $(a,b,c) \mapsto (a/b, c/b^2)$.
b. Let $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. Show that $U_2 \simeq V$ via $(a,b,c) \mapsto (a^2/c, ab/c, b^2/c)$.
c. Show that V is isomorphic to the affine toric variety U_σ for the cone $\sigma = \mathrm{Cone}(e_1, e_1 + 2e_2)$. Hint: Show that $S_\sigma \cap M$ is generated by $u_0 = e_2^*$, $u_1 = e_1^*$ and $u_2 = 2e_1^* - e_2^*$. Note that $u_0 + u_2 = 2u_1$.

In Lecture 4, we will show that $\mathbb{P}(1,1,2)$ is the abstract variety obtained by "gluing" two copies of $\mathbb{C}^2$ together with the affine toric variety U_σ from part c of Exercise 1.15. But before we can do this, we must first understand what "gluing" means.

§2. Abstract Varieties

The Definition of Manifold. To better understand the definition of abstract variety, we begin by recalling the definition of a C^∞ n-manifold. Such a manifold consists of a second countable Hausdorff topological space M together with an open cover U_α and homeomorphisms $f_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{R}^n$ is open, such that for every α, β, the composition

$$f_\beta \circ f_\alpha^{-1} : f_a(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.

It turns out that there is a simpler, though more sophisticated, way of giving this definition. We begin with an open set $V \subset \mathbb{R}^n$. The *sheaf* of C^∞ functions on V, denoted $\mathcal{O}_V^\infty$, is defined by assigning to each open set $U \subset V$ the $\mathbb{R}$-algebra

$$\mathcal{O}_V^\infty(U) = \{f : U \to \mathbb{R} \mid f \text{ is a } C^\infty \text{ function}\}. \tag{2.1}$$

More generally, given a topological space X, we say that $\mathcal{F}$ is *sheaf of* $\mathbb{R}$-*algebras* on X if for every open set $U \subset X$, there is an $\mathbb{R}$-algebra $\mathcal{F}(U)$ such that:

- If $V \subset U$ are open, then there is an $\mathbb{R}$-algebra homomorphism $r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$.*
- r_U^U is the identity and if $W \subset V \subset U$ are open, then $r_W^V \circ r_V^U = r_W^U$.*
- If $U = \bigcup_\alpha U_\alpha$, where U_α is open, then we have an exact sequence

$$0 \to \mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta),$$

where the second arrow is the map $\mathcal{F}(U) \to \mathcal{F}(U_\alpha)$ and the double arrows are the maps $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\alpha \cap U_\beta)$ and $\mathcal{F}(U_\beta) \to \mathcal{F}(U_\alpha \cap U_\beta)$.

Elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U, and when $V \subset U$, $r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ the *restriction map*. In the third bullet, exactness at $\mathcal{F}(U)$ means that sections of $\mathcal{F}(U)$ are determined locally, i.e., two sections are equal if and only if their restrictions to the U_α are equal. Exactness at $\prod_\alpha \mathcal{F}(U_\alpha)$ means that compatible sections patch, i.e., sections over the U_α which agree on their intersections come from a section over U.

Given a sheaf $\mathcal{F}$ and $U \subset X$ open, the sections of $\mathcal{F}$ over U can be denoted

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F}).$$

We will use the first of these in this lecture but will switch to the second for Lecture 9. When discussing sheaf cohomology in later lectures, we will use the third.

When $\mathcal{F}$ is a sheaf of $\mathbb{R}$-algebras on X, we call the pair $(X, \mathcal{F})$ a *ringed space over* $\mathbb{R}$. For example, when $V \subset \mathbb{R}^n$ is open, (2.1) gives a ringed space over $\mathbb{R}$ denoted $(V, \mathcal{O}_V^\infty)$.

Exercise 2.1. Complete the following definitions:

a. The *restriction* $\mathcal{F}\big|_U$ of a sheaf $\mathcal{F}$ on X to an open set $U \subset X$ is defined by ...
b. Ringed spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$ over $\mathbb{R}$ are *isomorphic* if there are a homeomorphism $\phi : X \to Y$ and, for $U \subset Y$ open, an $\mathbb{R}$-algebra isomorphism $\phi_U^\# : \mathcal{G}(U) \to \mathcal{F}(\phi^{-1}(U))$, such that ...

We can now reformulate the definition of C^∞ n-manifold.

Exercise 2.2. Let n be a positive integer and let $(M, \mathcal{O}_M)$ be a ringed space over $\mathbb{R}$. Assume that every point in M has a neighborhood U such that $(U, \mathcal{O}_M\big|_U)$ is isomorphic to $(V, \mathcal{O}_V^\infty)$ for some open subset $V \subset \mathbb{R}^n$. Prove that M has the structure of a C^∞ n-manifold.

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* These two bullets say that $\mathcal{F}$: **Open sets of** $X \to \mathbb{R}$**-Algebras** is a contravariant functor.

Exercise 2.3. Conversely, let M be a C^∞ n-manifold as defined at the beginning of the section.

a. Given $U \subset M$ open, define what it means for $f : U \to \mathbb{R}$ to be C^∞.
b. Use the definition given in part a to define the sheaf $\mathcal{O}_M$ of C^∞ functions on M and show that $(M, \mathcal{O}_M)$ is a ringed space over $\mathbb{R}$ which satisfies the condition of Exercise 2.2.

For a C^∞ n-manifold M, the sheaf $\mathcal{O}_M$ of Exercise 2.3 is called the *structure sheaf* of M.

Exercise 2.4. Give sheaf-theoretic definitions of a C^k n-manifold and a complex n-manifold.

The Structure Sheaf of an Affine Variety. We first show that some of the constructions for $\mathbb{C}^n$ given in Lecture 1 generalize to an arbitrary affine variety $V = \mathrm{Spec}(R)$.

Exercise 2.5. Let $V = \mathrm{Spec}(R)$ be an affine variety.

a. Given an ideal $I \subset R$, define $\mathbf{V}(I) \subset V$. Then prove that $\mathbf{V}(I)$ is a subvariety of V and that all subvarieties of V arise in this way.
b. Given a subvariety $W \subset V$, define $\mathbf{I}(W) \subset R$ and prove that $\mathbf{I}(W)$ is a radical ideal of R.
c. Prove the Nullstellensatz, i.e., that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$ for any ideal $I \subset R$.
d. Prove the Hilbert Basis Theorem, i.e., that any ideal $I \subset R$ can be written in the form $I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s \in R$.

In Lecture 1, we defined the Zariski open $V_f \subset V$ for any $f \in R$. Furthermore, when V is irreducible, we showed that $V_f = \mathrm{Spec}(R_f)$, where

$$R_f = \{g/f^m \in \mathbb{C}(V) \mid g \in R,\ m \geq 0\}$$

is the localization of R at f, as defined in Exercise 1.9 of Lecture 1.

Exercise 2.6. Show that the sets V_f form a basis of the Zariski topology of V.

The *structure sheaf* of an irreducible affine variety $V = \mathrm{Spec}(R)$ is the sheaf of $\mathbb{C}$-algebras in the Zariski topology defined as follows. Given a Zariski open $U \subset V$, a function $\phi : U \to \mathbb{C}$ is *regular* if for every $p \in V$, there is $f_p \in R$ such that $p \in V_{f_p} \subset U$ and $\phi\big|_{V_{f_p}} \in R_{f_p}$. Then

$$\mathcal{O}_V(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is a regular function}\}.$$

We will not show that $\mathcal{O}_V$ is a sheaf of $\mathbb{C}$-algebras—we refer the reader to [2] or [3] for the details of the proof. These references also show how to define $\mathcal{O}_V$ when V is not irreducible.

Exercise 2.7. Let $V = \mathbb{C}^2$ and set $U = \mathbb{C}^2 - \{(0,0)\}$. Show that $\mathcal{O}_V(U) = \mathbb{C}[x, y]$.

The structure sheaf $\mathcal{O}_V$ has two important properties.

Theorem 2.1. *Let $V = \mathrm{Spec}(R)$ be an irreducible affine variety.*

a. $\mathcal{O}_V(V) = R$.
b. *If $f \in R$, then $\mathcal{O}_V\big|_{V_f} = \mathcal{O}_{V_f}$.*

Proof. For part a, it suffices to show $\mathcal{O}_V(V) \subset R$. If $\phi : V \to \mathbb{C}$ is a morphism, then for each $p \in V$, there are $f_p, g_p \in R$ such that $\phi = g_p/f_p^{m_p}$ and $f_p(p) \neq 0$. Let $I = \langle f_p^{m_p} \mid p \in V \rangle \subset R$. It follows easily that $\mathbf{V}(I) = \emptyset$, so that by the Nullstellensatz, $\sqrt{I} = \mathbf{I}(\mathbf{V}(I)) = \mathbf{I}(\emptyset) = R$. Thus $1 \in R$, which implies that $1 = \sum_{p \in S} h_p f_p^{m_p}$ where $h_p \in R$ and $S \subset V$ is finite. Then

$$\phi = \sum_{p \in S} h_p f_p^{m_p}\, \phi = \sum_{p \in S} h_p\, g_p \in R.$$

For part b, let $U \subset V$ be Zariski open. If $\phi : U \to \mathbb{C}$ is a morphism, then for every $p \in U$, there is $p \in V_{f_p} \subset U$ such that $\phi \in R_{f_p}$. Now suppose in addition that $U \subset V_f$. If we regard f_p as an element of R_f, then one easily sees that

$$\phi \in R_{f_p} \subset R_{ff_p} = (R_f)_{f_p}.$$

Furthermore, $V_{f_p} \cap V_f = (V_f)_{f_p}$ shows that $p \in (V_f)_{f_p} \subset U \subset V_f$. By definition, this implies that $\phi \in \mathcal{O}_{V_f}(U)$, and part b now follows easily. □

Combining parts a and b of Theorem 2.1, we conclude that

$$\mathcal{O}_V(V_f) = \mathcal{O}_V\big|_{V_f}(V_f) = \mathcal{O}_{V_f}(V_f) = R_f$$

when $V = \operatorname{Spec}(R)$ and $f \in R$.

The Definition of Abstract Variety. We now give the main definition of this section.

Definition 2.2. *An* **abstract variety** $(X, \mathcal{O}_X)$ *is a ringed space over* $\mathbb{C}$ *where each* $p \in X$ *has a neighborhood* U *such that the restriction* $(U, \mathcal{O}_X\big|_U)$ *is isomorphic (as a ringed space over* $\mathbb{C}$*) to* $(V, \mathcal{O}_V)$ *for some affine variety* V*.*

Given an abstract variety $(X, \mathcal{O}_X)$, an open set $U \subset X$ is an *affine open* if $(U, \mathcal{O}_X\big|_U)$ is isomorphic (over $\mathbb{C}$) to the ringed space of an affine variety. The topology on X is called the *Zariski topology* since it restricts to the Zariski topology in each affine open subset.

Exercise 2.8. Let $(X, \mathcal{O}_X)$ be an abstract variety and let $U \subset X$ be Zariski open. Show that every section $\phi \in \mathcal{O}_X(U)$ gives a function $\phi : U \to \mathbb{C}$. We say that ϕ a *regular function* on U.

Exercise 2.9. Let $(X, \mathcal{O}_X)$ be an abstract variety and let $Y \subset X$ be Zariski closed. If $U_1 \subset Y$ is open, define $\mathcal{O}_Y(U_1)$ to be the set of all functions $\phi : U_1 \to \mathbb{C}$ such that for every $p \in U_1$, there is $U \subset X$ open and a regular function $\tilde{\phi} : U \to \mathbb{C}$ with $p \in U \cap Y \subset U_1$ and $\phi\big|_{U\cap Y} = \tilde{\phi}\big|_{U\cap Y}$.

a. Show that $U_1 \mapsto \mathcal{O}_Y(U_1)$ is a sheaf of $\mathbb{C}$-algebras on Y.
b. When $(X, \mathcal{O}_X)$ is an affine variety and $Y \subset X$ is a subvariety, prove that the sheaf defined in part a is precisely the sheaf of regular functions on Y
c. When $(X, \mathcal{O}_X)$ is an abstract variety and $Y \subset X$ is Zariski closed, prove that the sheaf of part a makes $(Y, \mathcal{O}_Y)$ into an abstract variety.

Given an abstract variety $(X, \mathcal{O}_X)$, we say that $Y \subset X$ is a *subvariety* if it is Zariski closed. The above exercise shows that Y inherits the structure of an abstract variety in a natural way. We define $(X, \mathcal{O}_X)$ to be *irreducible* if X is not the union of two proper subvarieties.

Exercise 2.10. Prove that an abstract variety is irreducible if and only if it is connected and every affine open subset is irreducible.

Let us show that $\mathbb{P}^n$ can be regarded as an abstract variety. In order to define the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$, we will use the field of rational functions $\mathbb{C}(\mathbb{P}^n)$ defined in Exercise 1.6 of this lecture. If $U \subset \mathbb{P}^n$ is Zariski open, then a function $\phi : U \to \mathbb{C}$ is *regular* if for each $p \in U$, there is $f/g \in \mathbb{C}(\mathbb{P}^n)$ such that $g(p) \neq 0$ and $\phi\big|_{U\cap \mathbf{V}(g)} = (f/g)\big|_{U\cap \mathbf{V}(g)}$. Then

$$\mathcal{O}_{\mathbb{P}^n}(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is a regular function}\}$$

defines a sheaf $\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$. We also have the affine open sets $U_i = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid a_i \neq 0\}$. In Exercise 1.11, we noted that if we regard $x_0/x_i, \ldots, x_{i-1}/x_i, x_{i-1}/x_i, \ldots, x_n/x_i$ as coordinates on $\mathbb{C}^n$, then the map $U_i \simeq \mathbb{C}^n$ induces an isomorphism

$$\mathbb{C}(\mathbb{P}^n) \simeq \mathbb{C}(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i-1}/x_i, \ldots, x_n/x_i).$$

Using this isomorphism, it is easy to see that $(U_i, \mathcal{O}_{\mathbb{P}^n}|_{U_i})$ is isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. This proves that $\mathbb{P}^n$ is an abstract variety. By Exercise 2.10, we see that $\mathbb{P}^n$ is irreducible.

As is customary, we often write an abstract variety $(X, \mathcal{O}_X)$ as simply X, and we will also drop the "abstract". Thus, "the variety X" is short for "the abstract variety $(X, \mathcal{O}_X)$".

Finally, a variety X also has a *classical topology*, which is the coarsest topology on X that agrees with the classical topology on every affine open subset of X. The structure sheaf $\mathcal{O}_X$ is not a sheaf in the classical topology. However, one can define the closely related sheaf $\mathcal{O}_X^{\mathrm{an}}$ of *analytic functions* on X, which is a sheaf in the classical topology. We call $(X, \mathcal{O}_X^{\mathrm{an}})$ the *complex analytic space* associated to the variety X. See [3, Appendix B] and [4] for more details.

The Function Field of an Irreducible Variety. If X is irreducible, then a *rational function* on X is a regular function $\phi : U \to \mathbb{C}$, where U is a nonempty Zariski open. Two rational functions are *equivalent* if they agree on some nonempty Zariski open, and the set of equivalence classes is denoted $\mathbb{C}(X)$. One can prove that $\mathbb{C}(X)$ is a field, called the *function field* of X.

Exercise 2.11. Let U be an affine open of an irreducible variety X. Prove that $\mathbb{C}(U) \simeq \mathbb{C}(X)$.

We say that $\phi \in \mathbb{C}(X)$ is *defined* at $p \in X$ there is a regular function $\phi' : U \to \mathbb{C}$ such that $p \in U$ and ϕ is equivalent to ϕ'. For $\phi \in \mathbb{C}(X)$, the set $\{p \in X \mid \phi \text{ is defined at } p\}$ is the largest Zariski open on which ϕ is defined.

The Local Ring of a Point. Given an irreducible variety X and a point $p \in X$, we define the *local ring* of X at p to be

$$\mathcal{O}_{X,p} = \{\phi \in \mathbb{C}(X) \mid \phi \text{ is defined at } p\}.$$

The key feature of $\mathcal{O}_{X,p}$ is described in the following exercise.

Exercise 2.12. Let $\mathcal{O}_{X,p}$ defined as above.
a. Show that $\mathfrak{m}_{X,p} = \{\phi \in \mathcal{O}_{X,p} \mid \phi(p) = 0\}$ is a maximal ideal of $\mathcal{O}_{X,p}$.
b. Given $\phi \in \mathcal{O}_{X,p}$, show that $\phi(p) \neq 0$ implies that $\phi^{-1} \in \mathcal{O}_{X,p}$.
c. Use part b to show that $\mathfrak{m}_{X,p}$ is the unique maximal ideal of $\mathcal{O}_{X,p}$.

In general, a commutative ring R with unit is called a *local ring* if it has a unique maximal ideal $\mathfrak{m}$. Thus $\mathcal{O}_{X,p}$ is a local ring by Exercise 2.12.

For an arbitrary variety X (not necessarily irreducible), one can define the local ring $\mathcal{O}_{X,p}$ to be the direct limit

$$\mathcal{O}_{X,p} = \varinjlim_{p \in U} \mathcal{O}_X(U),$$

where the limit is over all neighborhoods U of p. This is described in [2] and [3].

Morphisms. A *morphism* or *regular map* consists of a continuous map $\phi : X \to Y$ and, for each Zariski open $U \subset Y$, a $\mathbb{C}$-algebra homomorphism $\phi^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$, such that:

- $\phi^\#$ is compatible with restriction maps.

- For each $p \in X$, the map of local rings $\phi_p^{\#} : \mathcal{O}_{Y,\phi(p)} \to \mathcal{O}_{X,p}$ induced by $\phi^{\#}$ is a *local homomorphism*, meaning that $\mathfrak{m}_{Y,\phi(p)} = (\phi_p^{\#})^{-1}(\mathfrak{m}_{X,p})$.

A morphism $(\phi, \phi^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is usually written $\phi : X \to Y$.

An important results is that if $V = \mathrm{Spec}(R)$ and $W = \mathrm{Spec}(S)$ are irreducible affine varieties, then giving a morphism $\phi : V \to W$ is equivalent to giving a $\mathbb{C}$-algebra homomorphism $\phi^* : S \to R$. This is proved in [2] and [3].

Gluing Together Affine Varieties. We first observe that a variety X can be constructed by "gluing together" affine varieties along Zariski open subsets. Namely, Definition 2.2 implies that X has an affine open cover U_α, so that $f_\alpha : U_\alpha \simeq V_\alpha$, where V_α is an affine variety. Then, for any α, β, the set

$$V_{\alpha\beta} = f_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha$$

is Zariski open in V_α, and the map

$$g_{\alpha\beta} = f_\beta \circ f_\alpha^{-1} : V_{\alpha\beta} \to V_{\beta\alpha}$$

is an isomorphism of Zariski open subsets. Furthermore, these maps are compatible as follows:

- $g_{\alpha\alpha} = 1_{V_\alpha}$ for every α.
- $g_{\beta\gamma}\big|_{V_{\beta\alpha} \cap V_{\beta\gamma}} \circ g_{\alpha\beta}\big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}} = g_{\alpha\gamma}\big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}}$ for every α, β, γ.

We call these the *compatibility conditions*.

Conversely, suppose we have a collection

$$(\{V_\alpha\}_\alpha, \{V_{\alpha\beta}\}_{\alpha,\beta}, \{g_{\alpha\beta}\}_{\alpha,\beta}),$$

where each V_α is an affine variety, $V_{\alpha\beta} \subset V_\alpha$ is Zariski open, and the $g_{\alpha\beta} : V_{\alpha\beta} \simeq V_{\beta\alpha}$ are isomorphisms which satisfy the above compatibility conditions. Then we get the topological space

$$X = \coprod_\alpha V_\alpha / \sim$$

where $a \in V_\alpha$ is equivalent to $b \in V_\beta$ if $a \in V_{\alpha\beta}$ and $b = g_{\alpha\beta}(a)$. Furthermore, the structure sheaves $\mathcal{O}_{V_\alpha}$ patch to give a sheaf $\mathcal{O}_X$, and from here it is straightforward to prove that $(X, \mathcal{O}_X)$ is a variety with an affine open cover U_α such that $U_\alpha \simeq V_\alpha$ for every α. In this situation, we say that X is obtained from the V_α by *gluing* them together along the $V_{\alpha\beta}$ via the $g_{\alpha\beta}$.

Exercise 2.13. Let $V_0 = V_1 = \mathbb{C}$, $V_{01} = V_{10} = \mathbb{C} - \{0\}$ and $g_{01}(x) = g_{10}(x) = x^{-1}$. Prove that this data determines the variety $\mathbb{P}^1$.

Cartesian Products. If X and Y are varieties, then their cartesian product $X \times Y$ exists, though the definition is subtle. The difficulty is that the usual product topology on $X \times Y$ gives the wrong topology. Here is an example.

Exercise 2.14. Show that the product topology on $\mathbb{C} \times \mathbb{C}$, where we use the Zariski topology on each factor, is not the Zariski topology on $\mathbb{C}^2$.

To get the correct definition of cartesian product, we begin with affine varieties. Suppose that $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{C}^n$, with variables $x_1, \ldots, x_n$ and $W = \mathbf{V}(g_1, \ldots, g_t) \subset \mathbb{C}^m$, with variables $y_1, \ldots, y_m$. Also let R and S be the coordinate rings of V and W respectively.

Exercise 2.15. Let $V = \mathrm{Spec}(R)$ and $W = \mathrm{Spec}(S)$ be as above.

a. Show that $V \times W \subset \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$ is the affine variety $\mathbf{V}(f_1,\dots,f_s,g_1,\dots,g_t)$, where $f_i(x_1,\dots,x_n), g_j(y_1,\dots,y_m) \in \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_m]$.
b. Show that the coordinate ring of $V \times W$ is $R \otimes_{\mathbb{C}} S$.

Thus the cartesian product of $V = \operatorname{Spec}(R)$ and $W = \operatorname{Spec}(S)$ is $V \times W = \operatorname{Spec}(R \otimes_{\mathbb{C}} S)$.

Exercise 2.16. Suppose that $\sigma_1 \subset (N_1)_{\mathbb{R}}$ and $\sigma_2 \subset (N_2)_{\mathbb{R}}$ are strongly convex rational polyhedral cones. Let $N = N_1 \oplus N_2$.
a. Show that $\sigma_1 \times \sigma_2 \subset N_{\mathbb{R}}$ is a strongly convex polyhedral cone.
b. Show that $S_{\sigma_1 \times \sigma_2} \simeq S_{\sigma_1} \oplus S_{\sigma_2}$ as semigroups.
c. Show that $\mathbb{C}[S_{\sigma_1 \times \sigma_2}] \simeq \mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}]$ as $\mathbb{C}$-algebras.
d. Conclude that the affine toric variety $U_{\sigma_1 \times \sigma_2}$ is the cartesian product $U_{\sigma_1} \times U_{\sigma_2}$.

In the general case, we think of X as obtained by gluing together Zariski open subsets of U_α, and similarly Y comes from gluing together Zariski open subsets of U'_β. Then $X \times Y$ is constructed by gluing together the affine varieties $U_\alpha \times U'_\beta$ along suitable Zariski open subsets. We omit the details of the construction, which can be found in [2] and [3].

As an example, $\mathbb{P}^n \times \mathbb{P}^m$ can be constructed by this method. If $x_0,\dots,x_n$ are coordinates on $\mathbb{P}^n$ and $y_0,\dots,y_m$ are coordinates on $\mathbb{P}^m$, then one can show that $V \subset \mathbb{P}^n \times \mathbb{P}^m$ is Zariski closed if and only if $V = \mathbf{V}(f_1,\dots,f_s)$, where $f_i \in \mathbb{C}[x_0,\dots,x_n;y_0,\dots,y_m]$ is *bihomogeneous*, meaning that it is separately homogeneous in the x_i and in the y_j.

Finally, we should mention that if X and Y are varieties, then the classical topology on $X \times Y$ is the product of the classical topologies on X and Y respectively.

§3. Separated, Quasi-Compact, Complete, and Normal Varieties

Separatedness. Given any variety X, the *diagonal map* of X is the map $\Delta : X \to X \times X$ defined by $\Delta(p) = (p,p)$ for $p \in X$. Then X is *separated* if the image of the diagonal map $\Delta(X)$ is Zariski closed in $X \times X$. Here are some examples of separated and non-separated varieties.

Exercise 3.1. Consider the variety X constructed by identifying two copies of $\mathbb{C}$ along $\mathbb{C} - \{0\}$ (in the language of §2, this corresponds to $V_0 = V_1 = \mathbb{C}$, $V_{01} = V_{10} = \mathbb{C} - \{0\}$ and $g_{01}(x) = g_{10}(x) = x$). Show that X is not separated.

Exercise 3.2. Prove that $\mathbb{C}^n$ is separated by considering $\mathbf{V}(x_1 - y_1,\dots,x_n - y_n) \subset \mathbb{C}^n \times \mathbb{C}^n$.

Exercise 3.3. Prove that a subvariety of a separated variety is separated.

Combining Exercises 3.2 and 3.3, we see that affine varieties are always separated. In Lecture 4, we will see that $\mathbb{P}^n$ is separated since it is a toric variety. By Exercise 3.3, it follows that every projective variety is separated.

In [4], Serre characterized separatedness in terms of the classical topology as follows.

Theorem 3.1. *A variety X is separated if and only if it is Hausdorff in the classical topology.*

For example, this theorem makes it easy to see that the variety X of Exercise 3.1 is not Hausdorff (the two copies of the origin in X do not have disjoint neighborhoods). This also gives a quick proof of Exercise 3.2 since $\mathbb{C}^n$ is Hausdorff in the classical topology.

Here are some properties of separated varieties.

Proposition 3.2. *Let X be a separated variety.*

a. *If U and V are affine open subsets of X, then $U \cap V$ is also an affine open of X.*
b. *If $f, g : Y \to X$ is a morphism of varieties, then $\{y \in Y \mid f(y) = g(y)\}$ is a subvariety of Y.*

Proof. For part a, we note that $\Delta : X \to X \times X$ identifies $U \cap V$ with $\Delta(X) \cap (U \times V)$. We know that $\Delta(X)$ is Zariski closed in $X \times X$ by the definition of separated. It follows that $\Delta(X) \cap (U \times V)$ is Zariski closed in $U \times V$. But $U \times V$ is affine by Exercise 2.15, which implies that $\Delta(X) \cap (U \times V) \simeq U \cap V$ is also affine.

To prove part b, consider $\phi : Y \to X \times X$ defined by $\phi(y) = (f(y), g(y))$. Since

$$\{y \in Y \mid f(y) = g(y)\} = \phi^{-1}(\Delta(X)),$$

we see that $\{y \in Y \mid f(y) = g(y)\}$ is Zariski closed in Y since ϕ is continuous in the Zariski topology and (by separatedness) $\Delta(X)$ is Zariski closed in $X \times X$. □

When studying differentiable manifolds, one always assumes that the underlying topological space is Hausdorff. Similarly, in algebraic geometry, the varieties of interest are almost always separated. For this reason, we henceforth reserve the term *variety* for a separated variety. A non-separated variety will be called a *pre-variety*. The toric varieties defined in the next lecture will be all separated (though there have been some recent papers on toric pre-varieties).

Quasi-Compactness. We say that a variety X is *quasi-compact* if X is the union of finitely many affine open subsets. Any affine variety is quasi-compact. An more interesting example is $\mathbb{P}^n$, which is quasi-compact since it the union of the affine open subsets $U_i = \mathbb{P}^n - \mathbf{V}(x_i)$.

Here are the main properties of quasi-compact varieties.

Proposition 3.3. *Let X be a quasi-compact variety. Then:*

a. *Every subvariety of X is quasi-compact.*
b. *Every Zariski open subset of X is quasi-compact.*
c. *Every Zariski open cover of X has a finite subcover.*

Proof. Suppose that $X = U_1 \cup \cdots \cup U_r$ where U_i is an affine open subset of X. Then part a is obvious since $Y = (Y \cap U_1) \cup \cdots \cup (Y \cap U_r)$ where $Y \cap U_i$ is an affine open of Y.

For part b, $U = (U \cap U_1) \cup \cdots \cup (U \cap U_r)$ shows that we can assume that $X = \mathrm{Spec}(R)$ is affine. Then $X - U = \mathrm{V}(I)$ for some ideal $I \subset R$. The Hilbert basis theorem implies $I = \langle f_1, \ldots, f_s \rangle$, and it follows that $U = \bigcup_{i=1}^s X_{f_i}$.

Finally, we leave part c as Exercise 3.4 below. □

Exercise 3.4. Prove that every Zariski open cover of a quasi-compact variety has a finite subcover.

This exercise explains there the term "quasi-compact" comes from. As with separatedness, the varieties of interest to algebraic geometers are almost always quasi-compact. Hence, from now on, whenever we say *variety*, we will mean a separated quasi-compact abstract variety. The toric varieties defined in Lecture 4 will be quasi-compact (though one sometimes considers toric varieties coming from infinite fans, which are not quasi-compact).

Completeness and Properness. A variety X is *complete* if the following conditions hold:

- X is separated.
- X is quasi-compact.
- For every other variety Y, the projection map $X \times Y \to Y$ is closed, meaning that the projection of a Zariski closed set in $X \times Y$ is Zariski closed in Y.

Exercise 3.5. Prove that a subvariety of a complete variety is complete.

Exercise 3.6. Prove that the cartesian product of two complete varieties is complete.

Exercise 3.7. Prove that a variety X is complete if and only if for all $m \geq 1$, the projection map $X \times \mathbb{C}^m \to \mathbb{C}^m$ is closed.

The most basic example of a complete variety is $\mathbb{P}^n$.

Theorem 3.4. *$\mathbb{P}^n$ is complete.*

We will prove this in Lecture 6 when we consider the toric variety of a complete fan. For another proof, note that Theorem 6 of [1, Chapter 8, §5] implies that the projection map $\mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$ is closed. By Exercise 3.7, it follows that $\mathbb{P}^n$ is complete, and then any projective variety is complete by Exercise 3.5.

The completeness of $\mathbb{P}^n$ is closely related to elimination theory. To see why, suppose we have polynomials

$$f_1, \dots, f_s \in \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_m]$$

which are homogeneous in $x_0, \dots, x_n$. We can think of the f_i as homogeneneous polynomials in the x_i whose coefficients depend on the "parameters" y_j. Question: For which values of the parameters y_j do the equations

$$f_1 = \cdots = f_s = 0 \tag{3.1}$$

have a nontrivial solution in the x_i?

To answer this question, observe that (3.1) defines a subvariety $W \subset \mathbb{P}^n \times \mathbb{C}^m$, and the values of the y_j for which (3.1) has a nontrivial solution is the image of W under the projection map $\mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$. Since $\mathbb{P}^n$ is complete, this image is a variety in $\mathbb{C}^m$. In other words, there are polynomials $g_1, \dots, g_l \in \mathbb{C}[y_1, \dots, y_m]$ such that (3.1) has a nontrivial solution for the parameter values $y_j = b_j$ if and only if

$$g_1(b_1, \dots, b_m) = \cdots = g_l(b_1, \dots, b_m) = 0. \tag{3.2}$$

Chapter 8 of [1] gives an algorithm for finding the polynomials g_i. We say that (3.2) is obtained from (3.1) by *eliminating* the variables $x_0, \dots, x_n$. This is *projective elimination theory*.

Serre's paper [4] characterizes completeness in terms of the classical topology as follows.

Theorem 3.5. *A variety X is complete if and only if it is compact in the classical topology.*

This theorem and the Hopf fibration $S^{2n+1} \to \mathbb{P}^n$ give another proof that $\mathbb{P}^n$ is complete (see Exercise 1.2). In algebraic geometry, completeness is a very useful property. Here is a result which indicates some reasons why.

Theorem 3.6. *Let X be a complete variety.*

a. *If $\phi : X \to Y$ is a morphism, then its image $\phi(X)$ is a subvariety of Y.*
b. *If $\phi : \mathbb{C}^* \to X$ is a morphism, then ϕ extends uniquely to a morphism $\tilde{\phi} : \mathbb{C} \to X$.*
c. *X is affine if and only if X is a finite set of points.*
d. *If X is connected, then every morphism $\phi : X \to \mathbb{C}$ is constant.*

Part b of this theorem says that if X is complete and $\phi : \mathbb{C}^* \to X$ is a morphsim, then $\lim_{t \to 0} \phi(t)$ exists as a unique element of X whenever X is complete (in fact, $\lim_{t \to \infty} \phi(t)$ also exists, so that ϕ extends to a map $\mathbb{P}^1 \to X$). Part c says that affine varieties are very far from being complete since these concepts coincide only for finite sets of points.

Finally, completeness closely related to the idea of a *proper morphism* $\phi : X \to Y$. We will not give the definition here (we would need to define *fibered products* and *morphisms of finite type*). The reader should consult [2] and [3] for the full definition. Turning to Serre's classic paper [4] yet again, we can characterize properness in terms of the classical topology as follows.

Theorem 3.7. *A morphism $\phi : X \to Y$ is proper if and only if it is proper in the classical topology, meaning that $\phi^{-1}(C) \subset X$ is compact whenever $C \subset Y$ is compact.*

Exercise 3.8. Prove that X is complete if and only if $X \to \{\text{pt}\}$ is proper, where $\{\text{pt}\}$ is the variety consisting of a single point.

Normality. A variety X is *normal* if it is irreducible and the local ring $\mathcal{O}_{X,p}$ is integrally closed for every $p \in X$. In order to relate this to the definition of normal affine variety given in Lecture 1, we first generalize Exercise 3.2 of Lecture 2.

Exercise 3.9. Let R be an integral domain with field of fractions K. A subset $S \subset R$ is a *multiplicative subset* if $1 \in S$, $0 \notin S$, and S is closed under multiplication. Then the *localization* of R at S is $R_S = \{a/b \in K \mid a \in R, b \in S\}$.

a. Prove that R_S is the smallest subring of K containing R such that every $s \in S$ is invertible in R_S.
b. Prove that if R is integrally closed, then R_S is integrally closed.

We now show that for affine varieties, our two notions of normal coincide.

Proposition 3.8. *Let $V = \operatorname{Spec}(R)$ be an irreducible affine variety. The R is integrally closed if and only if the local ring $\mathcal{O}_{V,p}$ is integrally closed for all $p \in V$.*

Proof. We know from Lecture 1 that $p \in V$ corresponds to a maximal ideal $M \subset R$. Then $R - M$ is a multiplicative subset since maximal implies prime, and one easily shows that

$$\mathcal{O}_{V,p} = R_{R-M}.$$

If R is integrally closed, then Exercise 3.9 implies that $\mathcal{O}_{V,p}$ is also integrally closed. Conversely, if all of the $\mathcal{O}_{V,p}$ are integrally closed, then so is

$$\bigcap_{p \in V} \mathcal{O}_{V,p}$$

by Exercise 3.3 of Lecture 2. However, this intersection is precisely $\mathcal{O}_V(V)$, which equals R by part a of Theorem 2.1 of this lecture. It follows that R is integrally closed. □

Proposition 3.8 has a following immediate corollary.

Corollary 3.9. *An irreducible variety X is normal if and only if it is a union of affine varieties $U_\alpha = \operatorname{Spec}(R_\alpha)$ where R_α is integrally closed.*

§4. Gluing Affine Toric Varieties

In §2 we learned how to construct varieties by gluing together Zariski open subsets of affine varieties. For affine toric varieties, we can use the faces of a cone to specify the desired gluing.

The full details of this will be given in Lecture 4. Here, we will just indicate the basic ideas and give a simple example. To begin, suppose that $\sigma \subset N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone. Then, if τ is a face of σ, we get affine toric varieties $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ and $U_\tau = \operatorname{Spec}(\mathbb{C}[S_\tau])$. We first determine how these are related.

In Proposition 5.5 of Lecture 1, we showed that $\tau = H_u \cap \sigma$ for some $u \in S_\sigma$ and that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

Furthermore, by Proposition 2.3 of Lecture 2, this implies that the natural map $U_\tau \to U_\sigma$ is injective and that its image can be identified with the Zariski open subset $(U_\sigma)_{\chi^u}$. Thus

$$U_\tau \simeq (U_\sigma)_{\chi^u}. \tag{4.1}$$

Now suppose that σ and σ' are strongly convex rational polyhedral cones in $M_\mathbb{R}$ whose intersection $\tau = \sigma \cap \sigma'$ is a face of each. In Exercise 5.6 of Lecture 1, you showed how to find $u \in M$ such that

$$\begin{aligned} \tau &= H_u \cap \sigma,\ u \in S_\sigma \\ \tau &= H_{-u} \cap \sigma',\ -u \in S_{\sigma'}. \end{aligned} \tag{4.2}$$

Applying (4.1), we get

$$U_\sigma \supset (U_\sigma)_{\chi^u} \simeq U_\tau \simeq (U_{\sigma'})_{(\chi^u)^{-1}} \subset U_{\sigma'}. \tag{4.3}$$

As you can see, this is the perfect set-up for gluing U_σ and $U_{\sigma'}$ in such a way that U_τ becomes their intersection. Here is a simple example of what this looks like in practice.

Exercise 4.3. Let $N = \mathbb{Z}$ and set $\sigma = \text{Cone}(e_1)$, $\sigma' = \text{Cone}(-e_1)$. Note that $\tau = \{0\}$.

a. Show that we can pick $u = e_1$ in (4.2).

b. Show that (4.3) can be written as

$$\mathbb{C} = \text{Spec}(\mathbb{C}[x]) \supset \text{Spec}(\mathbb{C}[x])_x = \mathbb{C}^* = \text{Spec}(\mathbb{C}[x^{-1}])_{x^{-1}} \subset \text{Spec}(\mathbb{C}[x^{-1}]) = \mathbb{C}.$$

c. Show that this data is equivalent to the data of Exercise 2.13 of this lecture.

d. Conclude that $\mathbb{P}^1$ is the variety obtained by gluing U_σ and $U_{\sigma'}$ together along U_τ.

This exercise shows that we can give a purely "toric" construction of $\mathbb{P}^1$. Lecture 4 will use a combinatorial device called a *fan* to construct arbitrary toric varieties in the same way.

References

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