Lecture 7
Toric Surfaces

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This lecture will discuss toric surfaces and is based on portions of Sections 2.2, 2.5 and 2.6 of [2]. We will also discuss lattice polygons and the number 12, following [6].

§1. Singularities of Toric Surfaces

At the end of Lecture 5, we noted that a toric surface has at most finitely many singular points. These points correspond to the 2-dimesional cones of the fan and are at worst finite Abelian quotient singularities. We now study these singularities in more detail.

Cyclic Quotient Singularities. Let $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^2$ be a 2-dimensional strongly convex rational polyhedral cone in the plane, and let $v_1, v_2 \in N$ be its minimal generators. Since v_2 is primitive, we can choose a basis e_1, e_2 of N such that $v_2 = e_2$. Then $v_1 = me_1 + le_2$ for some $m, l \in \mathbb{Z}$. Note that m, l are relatively prime since v_1 is also primitive.

Exercise 1.1. Explain why $m \neq 0$. Also explain why we may assume $m > 0$, and in this case, show that if $N' \subset N$ is the sublattice generated by v_1, v_2, then $N/N' \simeq \mathbb{Z}/m\mathbb{Z}$.

By Lecture 5, $U_\sigma \simeq \mathbb{C}^2/G$, where $G = N/N'$. Exercise 1.1 shows that G is cyclic, so that U_σ has at worst cyclic quotient singularities. Hence this is true for all toric surfaces.

Normal Form of Two-Dimensional Cones in the Plane. We next find a normal form for the cone σ. So far, we have a basis e_1, e_2 of N such that the minimal generators of σ are $v_1 = me_1 + le_2$ and $v_2 = e_2$, where $m > 0$ is the index of N' in N and $\gcd(m, l) = 1$.

We get a unique choice of l as follows.

Exercise 1.2. As above, suppose that $v_1 = me_1 + le_2$, $v_2 = e_2$, where $m > 0$ and $\gcd(m, l) = 1$.

a. Let $r \in \mathbb{Z}$. Show that $e_1' = e_1 + re_2$ and e_2 is a basis of N.
b. Relative to the basis of part a, show that $v_1 = me_1' + (l - rm)e_2$.
c. Explain why there is a unique choice of r so that $v_1 = me_1' - ke_2$, where $0 \leq k < m$ and $\gcd(m, k) = 1$.

From here on, we will assume that a basis e_1, e_2 of N has been chosen so that the minimal generators of σ are $v_1 = me_1 - ke_2$, $v_2 = e_2$, where $0 \leq k < m$ and $\gcd(m, k) = 1$. We call m, k the *parameters* of σ.

Exercise 1.3. Let σ be a cone with parameters m, k. Prove that $\sigma^\vee$ has parameters m, $m - k$.

We can also show that the basis e_1, e_2 is uniquely determined.

Exercise 1.4. Let $\tilde{e}_1, \tilde{e}_2$ be another basis of N such that $v_1 = m\tilde{e}_1 - k\tilde{e}_2$ and $v_2 = \tilde{e}_2$, where $0 \leq k < m$ and $\gcd(m, k) = 1$. Prove that $\tilde{e}_i = e_i$ for $i = 1, 2$.

We call e_1, e_2 the *normalized basis* of N relative to σ when the above conditions are satisfied. For the rest of this section, fix a normalized basis e_1, e_2 and let e_1^*, e_2^* be the dual basis of M.

We next describe how the parameters m, k determine the action of $\mathbb{Z}/m\mathbb{Z} \simeq N/N'$ on $\mathbb{C}^2$.

Exercise 1.5. Let M' be the dual of N' and let $u_1, u_2 \in M'$ be dual to $v_1, v_2 \in N'$.

a. Prove that
$$\begin{aligned}\langle u_1, e_1\rangle &= 1/m\\ \langle u_2, e_1\rangle &= k/m\\ \langle u_1, e_2\rangle &= 0\\ \langle u_2, e_2\rangle &= 1.\end{aligned}$$

b. Let $\mu_m = \{\zeta \in \mathbb{C} \mid \zeta^m = 1\}$ be the group of mth roots of unity. Show that $\mu_m \simeq N/N'$ via $e^{2\pi i j/m} \mapsto je_1 + N'$.

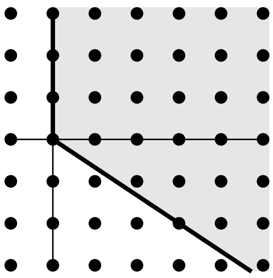
c. By Lecture 5, N/N' acts on $\mathbb{C}^2$ via $(v + N') \cdot \chi^{u'} = e^{2\pi i \langle u', v\rangle}\chi^{u'}$ for $u' \in \sigma^\vee \cap M'$ and $v + N' \in N/N'$. Using part b and the coordinates $x = \chi^{u_1}, y = \chi^{u_2}$ on $\mathbb{C}^2$, show that the action of $\zeta \in \mu_m \simeq N/N'$ is given by
$$\zeta \cdot (x, y) = (\zeta x, \zeta^k y).$$

Exercise 1.6. Let $v_1 = me_1 - ke_2$ and $v_2 = e_2$ be minimal generators of σ as above.

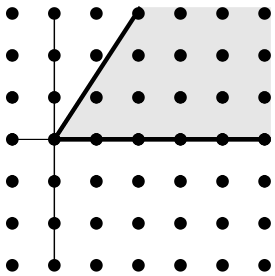
a. Let $u_1, u_2 \in M'$ be as in Exercise 1.5. Show that $u_1 = (1/m)e_1^*$ and $u_2 = (k/m)e_1^* + e_2^*$.

b. Conclude that $\sigma^\vee \subset M_{\mathbb{R}}$ is the cone generated by e_1^* and $ke_1^* + me_2^*$.

When $m = 3$ and $k = 2$, Exercise 1.6 gives the follow picture of σ and its dual $\sigma^\vee$:



The cone σ



The cone σ$^\vee$

In these pictures, we are using the bases e_1, e_2 of N (on the left) and e_1^*, e_2^* of M (on the right). Lattice points are indicated by dots.

Note that σ is regular if and only if $m = 1$, so that U_σ is smooth in this case. When $m > 1$, there are two cases when the singularity of U_σ is especially interesting and well-known.

The Cone over a Rational Normal Curve. Let σ be a cone with parameters $m > 1$ and $k = 1$ (so k is as small as possible). In this case, the action of $\zeta \in \mu_m$ on $\mathbb{C}^2$ is given by
$$\zeta \cdot (x, y) = (\zeta x, \zeta y),$$
where we are using the notation of Exercise 1.5.

Exercise 1.7. For the above μ_m-action, show that $\mathbb{C}[x, y]^{\mu_m} = \mathbb{C}[x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m]$.

This exercise shows that when $k = 1$, the coordinate ring of U_σ is

$$\mathbb{C}[U_\sigma] = \mathbb{C}[x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m] \subset \mathbb{C}[x, y]. \tag{1.1}$$

We can describe this geometrically as follows. The function $\mathbb{P}^1 \to \mathbb{P}^m$ defined by

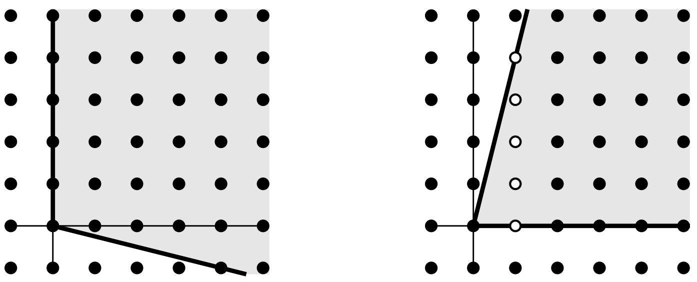
$$(x, y) \mapsto (x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m) \tag{1.2}$$

parametrizes the *rational normal curve of degree* m in $\mathbb{P}^m$. If we think of (1.2) as a map $\mathbb{C}^2 \to \mathbb{C}^{m+1}$, we get a surface called the *cone over the rational normal curve of degree* m. More precisely, Lecture 3 implies that the image of $\mathbb{P}^1 \to \mathbb{P}^m$ is a projective variety $C \subset \mathbb{P}^m$. Its ideal $\mathbf{I}(C) \subset \mathbb{C}[y_0, \ldots, y_m]$ is homogeneous, and the variety of $\mathbf{I}(C)$ in $\mathbb{C}^{m+1}$ is the *cone* over C.

Exercise 1.8. Let $S \subset \mathbb{C}^{m+1}$ be the image of the map (1.2).

a. Prove that S is the affine variety defined by $\mathbf{I}(C)$. (This is less trivial than it sounds since in general the image of a polynomial map is not a variety. The key point is that image in $\mathbb{P}^m$ *is* a variety.)
b. Prove that $\mathbb{C}[S] \simeq \mathbb{C}[x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m] \subset \mathbb{C}[x, y]$.
c. Use (1.1) to conclude that U_σ is the cone over the rational normal curve of degree m.

Finally, let us relate our discussion to the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$ of σ. When $m = 4$ and $k = 1$, σ and $\sigma^\vee$ have the following pictures:



The cone σ The cone $\sigma^\vee$

Here, the crucial observation is that the semigroup $\sigma^\vee \cap M$ is generated by $e_1^* + ie_2^*$ for $0 \le i \le 4$ (these are indicated by white dots in the above picture). This generalizes as follows.

Exercise 1.9. Let σ be a cone with $m > 1$ and $k = 1$, and let $U = \chi^{e_1^*}$ and $V = \chi^{e_2^*}$.

a. Prove that the semigroup $\sigma^\vee \cap M$ is generated by $e_1^* + ie_2^*$ for $0 \le i \le m$.
b. Conclude that $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[U, UV, \ldots, UV^m]$.
c. Our earlier discussion used $x = \chi^{u_1}$ and $y = \chi^{u_2}$, where u_1, u_2 are dual to v_1, v_2 as in Exercises 1.5 and 1.6. Use Exercise 1.6 to show that $U = x^m$ and $V = y/x$.
d. Show that $\mathbb{C}[U, UV, \ldots, UV^m] = \mathbb{C}[x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m]$ under the identifications of part c. Thus the toric description of $\mathbb{C}[U_\sigma]$ is consistent with (1.1).

Rational Double Points of Type A_k. Now let σ be a cone whose parameters m, k satisfy $k = m - 1$ (so k is as large as possible). For reasons which will soon become apparent, we will write $k + 1$ instead of m. Since $k \equiv -1 \bmod k + 1$, the action of $\zeta \in \mu_{k+1}$ on $\mathbb{C}^2$ is given by

$$\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y),$$

where $x = \chi^{u_1}$ and $y = \chi^{u_2}$ as in Exercise 1.5.

Exercise 1.10. Show that $\mathbb{C}[x, y]^{\mu_{k+1}} = \mathbb{C}[x^{k+1}, y^{k+1}, xy]$. Then show that

$$\mathbb{C}[x^{k+1}, y^{k+1}, xy] \simeq \mathbb{C}[X, Y, Z]/\langle Y^{k+1} - XZ \rangle$$

via the map defined by $X \mapsto x^{k+1}$, $Y \mapsto xy$ and $Z \mapsto y^{k+1}$.

The singularity of $\mathbf{V}(X^{k+1}+Y^2+Z^2) \subset \mathbb{C}^3$ at $0 \in \mathbb{C}^3$ is a *rational double point of type* A_k.

Exercise 1.11. Construct a coordinate change of $\mathbb{C}^3$ which takes $\mathbf{V}(X^{k+1}+Y^2+Z^2)$ into $\mathbf{V}(Y^{k+1}+XZ)$. This shows that when $m = k+1$, the singular point of U_σ is a rational double point of type A_k.

In general, a *rational double point* is the singularity of $\mathbb{C}^2/G$ at the origin, where G is a finite subgroup of $\mathrm{SU}(2,\mathbb{C})$. These subgroups are well-known and are related to the Dynkin diagrams A_k, D_k, E_6, E_7 and E_8. The groups corresponding to D_k, E_6, E_7 and E_8 are non-Abelian and hence do not arise in the toric context. This is why rational double points on toric surfaces are all of type A_k. See [1] for an excellent introduction to rational double points.

Finally, let us relate Exercise 1.10 to the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$.

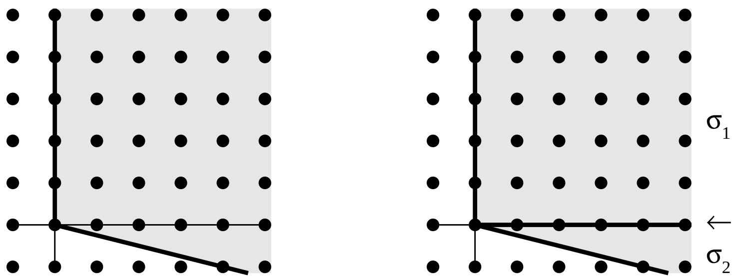
Exercise 1.12. Let σ be a cone with $m = k+1$, and let $U = \chi^{e_1^*}$ and $V = \chi^{e_2^*}$.
a. Prove that the semigroup $\sigma^\vee \cap M$ is generated by $e_1^*, e_1^*+e_2^*$, and $ke_1^*+(k+1)e_2^*$.
b. Conclude that $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[U, UV, U^kV^{k+1}]$.
c. Let $x = \chi^{u_1}$ and $y = \chi^{u_2}$ as in Exercises 1.5 and 1.6. Use Exercise 1.6 to show that $U = x^{k+1}$ and $V = y/x^k$.
d. Show that $\mathbb{C}[U, UV, U^kV^{k+1}] = \mathbb{C}[x^{k+1}, xy, y^{k+1}]$ under the identifications of part c. Thus the toric description of $\mathbb{C}[U_\sigma]$ coincides with what we found in Exercise 1.10.

Exercise 1.13. The special cases $k = 1$ and $m = k+1$ coincide when $m = 2$ and $k = 1$. Show that this is the affine toric variety $\mathbf{V}(xy - z^2) \subset \mathbb{C}^3$ introduced in Lecture 2.

§2. Resolution of Toric Surface Singularities

The singularities of the affine toric surface U_σ are easy to resolve. Here, $\sigma \subset N_\mathbb{R} \simeq \mathbb{R}^2$ is a 2-dimensional strongly convex rational polyhedral cone with parameters m, k defined in §1. We will assume $m > 1$, and we begin with the two special cases considered in §1.

Resolving the Cone over a Rational Normal Curve. If σ be a cone with $k = 1$, then U_σ is the cone over the rational normal curve of degree m. We will resolve the singularity of U_σ by subdividing σ into cones σ_1, σ_2 along the edge $\rho = \mathrm{Cone}(e_1)$. When $m = 4$ and $k = 1$, this gives the following picture.



The cone σ The cone σ subdivided into two cones

Let Δ consist of σ_1, σ_2 and their faces. Then one easily sees that Δ is a regular fan. This is obvious for σ_1 and follows from $k = 1$ for σ_2. Thus $X = X(\Delta)$ is a smooth toric surface. To see what X looks like, consider the following:

- $\mathrm{Star}(\rho) = \{\rho, \sigma_1, \sigma_2\}$, and the corresponding orbit closure $E = V(\rho)$ is isomorphic to $\mathbb{P}^1$.
- $X - E$ is the toric variety of $\Delta - \mathrm{Star}(\rho)$.
- But $\Delta - \mathrm{Star}(\rho)$ is the fan consisting of the proper faces of σ. The corresponding toric variety is $U_\sigma - \{x_\sigma\}$, where x_σ is the singular point of U_σ.

This tells us that $X - E = U_\sigma - \{x_\sigma\}$, so that we can regard X as a copy of U_σ where the singular point x_σ replaced by the *exceptional curve* $E \simeq \mathbb{P}^1$.

A better way to understand X is to realize that each cone of Δ lies in σ, which means that we get a morphism of toric varieties $\pi : X \to U_\sigma$. The above discussion shows that π induces an isomorphism $X - E \simeq U_\sigma - \{x_\sigma\}$ and maps E to the singular point x_σ. Note also that π is proper by Lecture 6. We say that $\pi : X \to U_\sigma$ is a *resolution of singularities* of U_σ.

A final comment is that the curve $E \subset X$ has a *self-intersection* $(E \cdot E)$. We will discuss self-intersections in §3. We will also see that the identity

$$e_2 + (me_1 - e_2) = me_1 \tag{2.1}$$

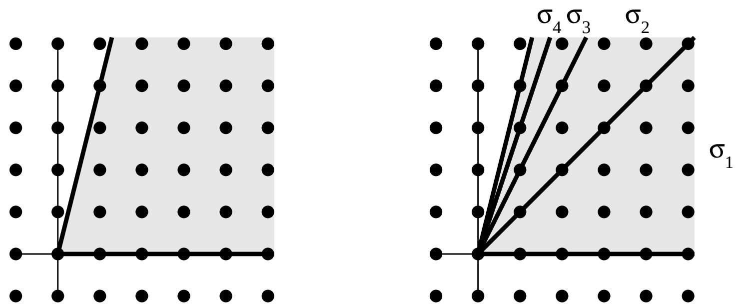
connecting the minimal generators of σ_1 and σ_2 implies that $(E \cdot E) = -m$. This is part of how the resolution determines the singularity it comes from.

Resolving Rational Double Points of Type A_k. We next suppose that σ is a cone with $m = k + 1$, so that U_σ has a rational double point of type A_k. In this case, we get the nicest resolution by picking a new basis for N.

Exercise 2.1. Let σ be a cone with $m = k + 1$ and let e_1, e_2 be the normalized basis of N relative to σ. Set $\tilde{e}_1 = e_2$ and $\tilde{e}_2 = e_1 - e_2$. Show that the minimal generators of σ are given by

$$v_1 = \tilde{e}_1 + (k+1)\tilde{e}_2, \quad v_2 = \tilde{e}_1.$$

We will resolve the singularity of U_σ by subdividing σ into $k + 1$ cones $\sigma_1, \ldots, \sigma_{k+1}$ by introducing the edges $\rho_j = \mathrm{Cone}(\tilde{e}_1 + j\tilde{e}_2)$ for $j = 1, \ldots, k$. When $m = 4$ and $k = 3$, we get the following picture (relative to the basis $\tilde{e}_1, \tilde{e}_2$):



The cone σ The subdivision of σ

Then let Δ be the fan consisting of the σ_i and their faces. In the following exercise, you will show how $X = X(\Delta)$ resolves a rational double point of type A_k.

Exercise 2.2. Let σ_i, ρ_i and Δ be as above.

a. Show that Δ is a regular fan, so that X is a smooth toric surface.
b. Show that $E_i = V(\rho_i) \simeq \mathbb{P}^1$. Let $E = E_1 \cup \cdots \cup E_k$.
c. Show that $X - E$ is the toric variety coming from the fan $\Delta - (\mathrm{Star}(\rho_1) \cup \cdots \cup \mathrm{Star}(\rho_k))$.
d. Show that the fan of part d is the fan consisting of all proper faces of σ. Conclude that $X - E$ is isomorphic to $U_\sigma - \{x_\sigma\}$, where x_σ is the singular point of U_σ.
e. Show that there is a proper toric morphism $\pi : X \to U_\sigma$ which maps E to x_σ and induces an isomorphism $X - E \simeq U_\sigma - \{x_\sigma\}$.

Exercise 2.3. Using the notation of Exercise 2.2, let $E_i = V(\rho_i) \simeq \mathbb{P}^1$ and $E = E_1 \cup \cdots \cup E_k$. Draw the graph with vertices $\{E_1, \ldots, E_k\}$, where E_i and E_j are linked by an edge if and only if $E_i \cap E_j \neq \emptyset$. The graph you get should be the Dynkin diagram for A_k.

Note also that the minimal generators of σ_j are $\tilde{e}_1 + (j-1)\tilde{e}_2$ and $\tilde{e}_1 + j\tilde{e}_2$. It follows that the generators of σ_j and σ_{j+1} satisfy the relation

$$(\tilde{e}_1 + (j-1)\tilde{e}_2) + (\tilde{e}_1 + (j+1)\tilde{e}_2) = 2(\tilde{e}_1 + j\tilde{e}_2). \tag{2.2}$$

As we will see in §3, this implies that E_j has self-intersection $(E_j \cdot E_j) = -2$. Rational double points of type A_k are characterized by having a resolution whose exceptional components have graph A_k and self-intersections all equal to -2.

A final observation concerns the relation between regular subdivisions and irreducible elements. If you compare the illustrations on pages 3 and 5, especially the pictures on the right hand side, you will see that when $m = k+1$, the edges of the regular subdivision of σ are generated by the irreducible elements of $\sigma \cap N - \{0\}$. Here is another case when this is true.

Exercise 2.4. Let σ satisfy $k = 1$. Prove that the edges of the regular subdivision of σ are generated by the irreducible elements of $\sigma \cap N - \{0\}$. Hint: Look at the picture on page 4. Then put the cone $\sigma^\vee$ of Exercise 1.12 into normal form.

In the next section, we will see that the relation between regular subdivisions and irreducible elements will give an elegant description of the resolution of a toric surface.

Resolving General Toric Surfaces. We can now prove the existence of a resolution of singularities $\pi : X' \to X$ for an arbitrary toric surface X. Suppose $X = X(\Delta)$, where Δ is a fan in $N_\mathbb{R} \simeq \mathbb{R}^2$. Let $\sigma_1, \ldots, \sigma_m$ be the cones of $\Delta(2)$. We can assume $m > 0$ since otherwise X is already smooth. For each i, let $N_i \subset N$ be the sublattice generated by the minimal generators of σ_i. We will use

$$s(\Delta) = \sum_{i=1}^{m} ([N : N_i] - 1)$$

as a measure of how singular X is. This is justified because $[N : N_i] \geq 1$, with equality if and only if σ_i is regular. It follows that $s(\Delta) \geq 0$, with equality if and only if Δ is regular. We call $[N : N_i]$ the *multiplicity* of σ_i.

We claim that if $s(\Delta) > 0$, then Δ has a refinement Δ_0, obtained by subdividing one of the cones of $\Delta(2)$, such that $s(\Delta_0) < s(\Delta)$. To construct Δ', pick a cone σ_i with $m = [N : N_i] > 1$ (σ_i exists because $s(\Delta) > 0$). Then let e_1, e_2 be a normalized basis of N relative to σ_i. Thus the minimal generators of σ_i are $v_1 = me_1 - ke_2$, $v_2 = e_2$, where $0 \leq k < m$ and $\gcd(m,k) = 1$. Now, as in the case $k = 1$, subdivide σ_i by inserting the edge $\rho = \mathrm{Cone}(e_1)$. This gives cones $\sigma_i' = \mathrm{Cone}(e_1, v_1)$ and $\sigma_i'' = \mathrm{Cone}(e_1, v_2)$. Since $v_2 = e_2$, we see that σ_i'' is regular. As for σ_i', we can analyze it as follows.

Exercise 2.5. Let $\sigma_i' = \mathrm{Cone}(e_1, v_1) = \mathrm{Cone}(e_1, me_1 - ke_2)$ be as above, and write $m = a_1 k - k_1$ where $0 \leq k_1 < k$. Prove that the parameters of σ_i' are k, k_1.

Now let Δ_0 the the fan obtained from Δ by replacing σ_i with σ_i', σ_i'', and ρ. We claim that

$$s(\Delta_0) < s(\Delta).$$

This follows because σ_i has multiplicity m while σ_i' and σ_i'' have multiplicities $k < m$ (by Execise 2.5) and 1 respectively.

If we repeat this process as many times as necessary, we eventually obtain a subdivison Δ' of Δ such that Δ' is regular. This gives a smooth toric variety $X' = X(\Delta')$ and a toric morphism $\pi : X' \to X$ with the following properties:

a. π is proper since Δ and Δ' have the same support.
b. Let $x_1, \dots, x_s$ be the singular points of X and set $E = \pi^{-1}(\{x_1, \dots, x_s\}) \subset X'$. Then:
  - E is a union of 1-dimensional orbit closures in X', each isomorphic to $\mathbb{P}^1$.
  - π induces an isomorphism $X' - E \simeq X - \{x_1, \dots, x_s\}$.

Thus $\pi : X' \to X$ is a resolution of singularities of X.

Hirzebruch-Jung Continued Fractions. When $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^2$ is a 2-dimensional rational polyhedral cone with parameters m, k, we can describe the resolution of U_σ using the *Hirzebruch-Jung* continued fraction

$$\frac{m}{k} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_r}}}, \tag{2.3}$$

where the $a_i \geq 2$ are integers. This is explained in the following exercises.

Exercise 2.6. Given relatively prime integers $0 \leq k < m$, the *modified Euclidean algorithm* considers successive divisions

$$\begin{aligned} m &= a_1 k - k_1, & 0 \leq k_1 < m \\ k &= a_2 k_1 - k_2, & 0 \leq k_2 < k_1 \\ &\vdots & \vdots \end{aligned}$$

a. Show that $a_i \geq 2$ for all $i \geq 1$.
b. Show that eventually $k_{r-1} = 1$ and $k_r = 0$. Thus the algorithm terminates.
c. Let r be as in part b. Show that r and $a_1, \dots, a_r$ have the same meaning as in (2.3). Hint: $m = a_1 k - k_1$ implies

$$\frac{m}{k} = a_1 - \frac{k_1}{k} = a_1 - \frac{1}{k/k_1}.$$

Exercise 2.7. Given (2.3), define p_i, q_i for $i \geq 0$ by setting

$$p_0 = 1, \quad q_0 = 0; \quad p_1 = a_1, \quad q_1 = 1$$

and, for $i \geq 1$,

$$\begin{aligned} p_{i+1} &= a_{i+1} p_i - p_{i-1} \\ q_{i+1} &= a_{i+1} q_i - q_{i-1}. \end{aligned}$$

a. Show that $p_{i+1} q_i - q_{i+1} p_i = -1$ for all $i \geq 0$.
b. Show that $p_{i+1} > p_i$ and $q_{i+1} > q_i$ for all $i \geq 0$.
c. Show that $\gcd(p_i, q_i) = 1$ for all $i \geq 0$.
d. Assume that for some $1 \leq i < r$, we have

$$\frac{p_i - p_{i-1} x}{q_i - q_{i-1} x} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_i - x}}}.$$

Prove that the same identity holds for $i + 1$. Hint: Let $x = 1/(a_{i+1} - y)$.

Setting $x = 0$ in part d of Exercise 2.7, we obtain

$$\frac{p_i}{q_i} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_i}}}.$$

Note that p_i/q_i is in lowest terms by part c of Exercise 2.7. We call p_i/q_i the ith *partial quotient* of m/k. Note also that $p_r/q_r = m/k$.

Exercise 2.8. Now let σ be a cone with parameters m, k and let e_1, e_2 be a normalized basis for N relative to σ. Thus

$$\sigma = \mathrm{Cone}(me_1 - ke_2, e_2).$$

Let $\tilde{v}_i = p_{i-1}e_1 - q_{i-1}e_2$ for $1 \le i \le r+1$, where p_i, q_i are from Exercise 2.7. Thus $\tilde{v}_1 = e_1$ and $\tilde{v}_{r+1} = me_1 - ke_2$. Also let $\tilde{v}_0 = e_2$. Then consider the cones

$$\sigma_i = \mathrm{Cone}(\tilde{v}_i, \tilde{v}_{i+1}), \quad i = 0, \ldots, r.$$

The goal is to show that $\sigma_0, \ldots, \sigma_r$ and their faces form a regular fan Δ' whose support is σ.

a. Prove that σ_i is regular. Hint: Part a of Exercise 2.7.
b. Prove that $\sigma_{i-1} \cap \sigma_i = \mathrm{Cone}(\tilde{v}_i)$. Hint: Compare slopes using part a of Exercise 2.7.
c. Show that $\sigma = \sigma_0 \cup \cdots \cup \sigma_r$, and conclude that Δ' has the desired properties.

If we let $X' = X(\Delta')$, where Δ' is the fan of Exercise 2.8, then our previous arguments show that we get a toric morphism

$$\pi : X' \to U_\sigma \tag{2.4}$$

which is a resolution of singularities. We get an intrinsic version of this resolution as follows.

Exercise 2.9. Let σ be a cone with parameters m, k and let $\tilde{v}_i$, $0 \le i \le r+1$ be as in Exercise 2.8. Then consider $S = \{\tilde{v}_0, \ldots, \tilde{v}_{r+1}\} \subset \sigma \cap N - \{0\}$.

a. Show that S generates $\sigma \cap N$ as a semigroup. Hint: Use Exercise 2.8.
b. If $\tilde{v}_i \in S$ is not irreducible, show that it can be expressed as a non-negative integer linear combination of the elements of $S - \{\tilde{v}_i\}$.
c. Show that every element of S is irreducible. Hint: Use part b together with part b of Exercise 2.7.
d. Conclude that S is the set of irreducible elements of $\sigma \cap N - \{0\}$.

Comparing the two previous exercises, we see that the minimal generators of the cones of Exercise 2.8 are the irreducible elements of $\sigma \cap N - \{0\}$. Thus, from an intrinsic point of view, the toric morphism (2.4) is obtained by subdividing the cone σ into subcones whose edges are the irreducible elements of $\sigma \cap N - \{0\}$.

We can also think of irreducible elements geometrically as follows.

Exercise 2.10. Let σ be as above. Show that the irreducible elements of $\sigma \cap N - \{0\}$ are the vertices of the convex hull $\mathrm{Conv}(\sigma \cap N - \{0\})$.

The $\tilde{v}_i \in \sigma \cap N - \{0\}$ from Exercise 2.8 have one further interesting property.

Exercise 2.11. Let $\tilde{v}_0, \ldots, \tilde{v}_{r+1}$ be as in Exercise 2.8. Use Exercise 2.7 to prove that

$$a_i \tilde{v}_i = \tilde{v}_{i+1} + \tilde{v}_{i-1}, \quad 1 \le i \le r. \tag{2.5}$$

This has the following consequence for the resolution (2.4). The edges of the fan Δ' are $\rho_i = \mathrm{Cone}(\tilde{v}_i)$. For $1 \le i \le r$, ρ_i is in the relative interior of σ, which means that the corresponding orbit closure $E_i = V(\rho_i) \subset X'$ is isomorphic to $\mathbb{P}^1$. In §3, we will see that (2.5) then implies that E_i has self-intersection $(E_i \cdot E_i) = -a_i$ for $1 \le i \le r$. Thus the continued fraction (2.3) determines everything about the exceptional divisor $E = E_1 \cup \cdots \cup E_r$ of $\pi : X' \to U_\sigma$.

Finally, we observe that the above description of irreducible elements can be used to give explicit generators for the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$.

Exercise 2.12. Let $\sigma \subset N_\mathbb{R} \simeq \mathbb{R}^2$ be a cone with parameters m, k, and let e_1, e_2 be a normalized basis of N relative to σ. We saw in Exercise 1.3 that $\sigma^\vee \subset M_\mathbb{R}$ has parameters m, $m-k$. Let $U = \chi^{e_1^*}$ and $V = \chi^{e_2^*}$, and let the partial quotients of $m/(m-k)$ be $P_1/S_1, \ldots, P_s/Q_s$.

a. Show that $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[U, UV, U^{P_1-Q_1}V^{P_1}, \ldots, U^{P_s-Q_s}V^{P_s}]$.
b. Use (3.2) from Lecture 5 to show that the dimension of the Zariski tangent space at the singular point of U_σ is $s+2$.
c. When $k=1$, show that $s = m-1$ and $P_i = Q_i + 1 = i+1$ for $1 \le i \le s$. Conclude that the description of part a agrees with Exercise 1.9
d. When $m = k+1$, show that $s = 1$, $P_1 = k+1$ and $Q_1 = 1$. Conclude that the description of part a agrees with Exercise 1.12.

There is a nice relation between the continued fraction expansions of m/k and $m/(m-k)$. This is discussed in [5, p. 29].

Reversing Orientation. The attentive reader may have noticed that the parameters m, k of a cone σ depend on the order in which we list the minimal generators v_1, v_2. In other words, m and k depend on the *orientation* of σ. To see what happens when we reverse the orientation, assume $m > 1$ and let $0 \le k' < m$ be the multiplicative inverse of k modulo m, i.e., $kk' \equiv 1 \bmod m$. We know that k' exists since $\gcd(m,k) = 1$.

Exercise 2.13. Let σ be a cone with minimal generators $v_1 = me_1 - ke_2$, $v_2 = e_2$, where $0 \le k < m$ and $\gcd(m,k) = 1$. Let k' be as above, and pick $m' \in \mathbb{Z}$ so that $mm' + kk' = 1$. Set

$$\tilde{e}_1 = k'e_1 + m'e_2$$
$$\tilde{e}_2 = me_1 - ke_2.$$

a. Show that $\tilde{e}_1, \tilde{e}_2$ form a basis of N.
b. Show that $v_2 = m\tilde{e}_1 - k'\tilde{e}_2$ and $v_1 = \tilde{e}_2$.

Exercise 2.13 shows that if σ has parameters m, k for v_1, v_2, then its parameters for v_2, v_1 are m, k'. Thus reversing the orientation of σ preserves m and interchanges k and k'.

This has a nice consequence for the continued fractions of m/k and m/k'. Write (2.3) as

$$\frac{m}{k} = [a_1, \ldots, a_r].$$

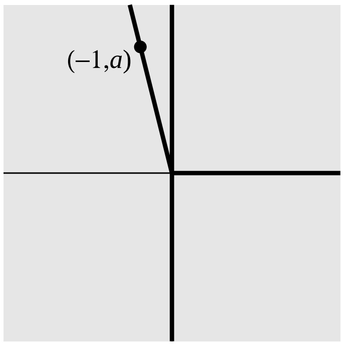
If σ is an oriented cone with parameters m, k, then (2.5) tells us that the a_i are determined by the irreducible elements of $\sigma \cap N - \{0\}$. When we reverse the orientation, we change k to k' and traverse the irreducible elements in reverse order. It follows that

$$\frac{m}{k'} = [a_r, \ldots, a_1].$$

Finally, suppose that σ_1 and σ_2 are oriented cones with parameters m_i, k_i, $i = 1, 2$. Then the affine toric varieties U_{σ_1} and U_{σ_2} are isomorphic as varieties $\Leftrightarrow$ if $m_1 = m_2$ and either $k_1 = k_2$ or $k_1k_2 \equiv \bmod m_1$. The $\Leftarrow$ direction follows from the above discussion, while the other direction is more difficult since we are not assuming that the isomorphism $U_{\sigma_1} \simeq U_{\sigma_2}$ is toric. Here, one needs the fact that such an isomorphism induces an isomorphism of minimal resolutions constructed in (2.4). Then the desired result follows by studying how the isomorphism acts on the exceptional divisor. Details of this argument, together with other facts about Hirzebruch-Jung continued fractions and the singularities of U_σ, can be found in [5, 1.6] and in the references given on page 134 of [2].

§3. Classification of Smooth Complete Toric Surfaces

Hirzebruch Surfaces. We begin our discussion of smooth complete toric surfaces by describing the *Hirzebruch surface* $\mathbb{F}_a$, which is the toric variey of the complete fan whose edges are generated by e_1, $\pm e_2$ and $-e_1 + ae_2$ for $a > 0$. The fan for $\mathbb{F}_a$ is pictured as follows:



It is an easy exercise to check that this fan is regular, so that $\mathbb{F}_a$ is a smooth complete toric surface. We will soon see that the $\mathbb{F}_a$, together with $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, are the basic building blocks of all smooth complete toric surfaces.

The Classification Theorem. Let $X = X(\Delta)$ be a smooth complete toric surface. If σ is a cone in $\Delta(2)$ with minimal generators v_1, v_2, then the fan Δ' obtained by subdividing σ along the edge generated by $v_1 + v_2$ is again regular. We say that Δ' is obtained from Δ by *stellar subdivision*. In Lecture 6, we saw that $X' = X(\Delta')$ is the blow-up of X at the point x_σ corresponding to σ. Note that X' is again a complete smooth toric surface.

Exercise 3.1. Show that $\mathbb{F}_1$ is isomorphic to the blow-up of $\mathbb{P}^2$ at a point.

The classification theorem for smooth complete toric surfaces goes as follows.

Theorem 3.1. *All smooth complete toric surfaces arise from $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and the Hirzebruch surfaces $\mathbb{F}_a$ for $a \geq 2$ by successive blow-ups corresponding to stellar subdivisions.*

Before give the proof, we need to do some preliminary work. Suppose that $X = X(\Delta)$ for some complete regular fan Δ, and let the minimal generators of the cones of Δ be $v_0, \ldots, v_d = v_0$, listed in clockwise from v_0. Then the 2-dimensional cones of Δ are generated by successive pairs. We will write the indices modulo d.

Exercise 3.2. Prove that there are integers a_i such that

$$v_{i-1} + v_{i+1} = a_i v_i \tag{3.1}$$

for all i. Hint: v_{i-1}, v_i and v_i, v_{i-1} are bases of N. Also remember that v_{i-1} and v_{i+1} lie on opposite sides of the line generated by v_i.

The following lemma will play a key role in the proof of Theorem 3.1.

Lemma 3.2. *Let Δ' be a regular subdivision of a regular cone σ. The Δ' is obtained from σ by a sequence of stellar subdivisions.*

Proof. Let the cones of Δ' have minimal generators $v_1 = \tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_r, \tilde{v}_{r+1} = v_2$, listed clockwise from v_1. We proceed by induction on r, the case $r = 0$ being obvious.

If for some $1 \leq i \leq r$ we have $\tilde{v}_{i-1} + \tilde{v}_{i+1} = \tilde{v}_i$, then Δ' comes from a stellar subdivision of a regular fan Δ'' which also subdivides σ. Then we are done by our inductive hypothesis.

On the other hand, if $\tilde{v}_{i-1} + \tilde{v}_{i+1} \neq \tilde{v}_i$ for all i, then (3.1) implies $\tilde{v}_{i-1} + \tilde{v}_{i+1} = a_i\tilde{v}_i$ for some $a_i \neq 1$. We also know that $a_i > 0$ since σ is strongly convex, and it follows that $a_i \geq 2$ for $1 \leq i \leq r$. Now consider the Hirzebruch-Jung continued fraction

$$\frac{m}{k} = [a_1, \ldots, a_r].$$

The results of §2, especially Exercise 2.7, imply that σ has parameters m, k (see Exercise 3.4 below). Since $0 < k < m$, we have $m \geq 2$. This is impossible since σ is regular. □

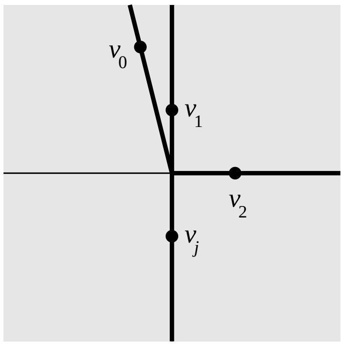
Exercise 3.3. In the situation of Lemma 3.2, assume $a_i \geq 2$ and let p_i/q_i be the partial quotients of $m/k = [a_1, \ldots, a_r]$.

a. Prove that $\tilde{v}_i = p_{i-1}\tilde{v}_1 - q_{i-1}\tilde{v}_0$ for $i = 1, \ldots, r+1$.
b. Conclude that $v_2 = m\tilde{v}_1 - k\tilde{v}_0$ and that σ has parameters m, k.

Proof of Theorem 3.1. Let Δ be a complete regular fan with minimal edge generators $v_0, \ldots, v_d = v_0$ as above. We also know that $v_{i-1} + v_{i+1} = a_i v_i$ for some $a_i \in \mathbb{Z}$. If some $a_i = 1$, the Δ is the stellar subdivision of a fan with fewer cones. By induction, we may assume that

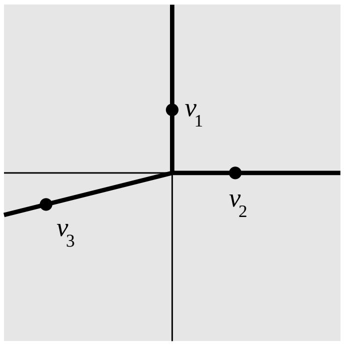
$$v_{i-1} + v_{i+1} = a_i v_i, \quad a_i \neq 1 \text{ for all } i. \tag{3.2}$$

First suppose that $v_i = -v_j$ for some i, j. Relabeling, we may assume $v_1 = -v_j$. Using the basis v_1, v_2, we get the following picture:



Since v_2, v_j generate a regular cone, Lemma 3.2 and (3.2) imply that $v_j = v_3 = -v_1$. Also note that $v_0 + v_2 = a_1 v_1$ and $a_1 \neq 1$. Then $v_0 = -v_2 + a_1 v_1$. The cones generated by v_0, v_1 and $v_0, v_3 = -v_1$ are regular. Then Lemma 3.2 and (3.2) show that v_0, v_1, v_2, v_3 are the only minimal edge generators of Δ. From here, one easily sees that $X = X(\Delta)$ is either $\mathbb{P}^1 \times \mathbb{P}^1$ (if $a_1 = 0$) or $\mathbb{F}_a$ (if $a = |a_1| \neq 0$).

To prove the theorem in general, we will use the basis v_1, v_2 of N. By the previous case, we can assume $v_3 \neq -v_1$. Then, relative to our chosen basis, v_3 must lie in the interior of the third or fourth quadrant. First suppose it lies in the third quadrant, as in the following picture:



We know that $v_1 + v_3 = a_2 v_2$, and $a_2 < 0$ since v_3 is in the third quadrant. Let σ be the oriented cone generated by v_3, v_1. If $a_2 = -1$, then σ is regular, so that $v_3 = v_0$ by Lemma 3.2 and (3.2). This implies $v_0 = -v_1 - v_2$, and $X = \mathbb{P}^2$ follows. On the other hand, suppose that $a_2 < -1$. Here, we put $m = -a_2$ and let $e_1 = -v_2$, $e_2 = v_1$. Then

$$v_3 = me_1 - e_2, \quad v_1 = e_2.$$

This shows that σ has parameters m and $k = 1$. Now observe that $v_3, v_4, \ldots, v_d = v_0, v_1$ lie in σ and give a regular subdivision. If we apply the argument of Lemma 3.2 and Exercise 3.3 to this situation, we see that $m/1 = [a_d, \ldots, a_4]$ (can you explain why the a_i's are ordered this way?). But $m/1 = m = [m]$, which (by uniqueness of continued fraction expansions) implies that $d = 4$ and $a_4 = -a_2$. Then the indices work modulo 4, so that

$$-a_2 v_0 = a_4 v_0 = a_4 v_4 = v_3 + v_5 = v_3 + v_1 = a_2 v_2.$$

Hence $v_0 = -v_2$, which puts us in the previous case.

The final case to consider is when v_3 lies in the fourth quadrant. Let v_j, $j \geq 3$, be the last cone in the fourth quadrant, as measured clockwise from v_2. We may assume $v_j \neq -v_1$. Then $v_j = av_2 - bv_1$ for $a, b > 0$. Our choice of v_j implies that $v_{j+1} = -cv_2 + dv_1$ for $c > 0$. If $d = 0$, then $v_{j+1} = -v_2$ (remember v_{j+1} is primitive), which puts us in a case already considered.

We will complete the proof by showing that $d \neq 0$ leads to a contradiction. The key point is that v_j, v_{j+1} span a smooth cone, so that

$$\det \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = ad - bc = \pm 1. \tag{3.3}$$

We know that $a, b, c, d \in \mathbb{Z}$ and $a, b, c > 0$. If $d < 0$, then $ad - bc < -1$, contradicting (3.3). If $d > 0$, then v_j (resp. v_{j+1}) lies in the fourth (resp. second) quadrant. Since the cone they span goes clockwise from v_j to v_{j+1}, it follows that $v_j + v_{j+1}$ lies in the third quadrant. Then

$$v_j + v_{j+1} = (av_2 - bv_1) + (-cv_2 + dv_1) = (a - c)v_2 + (d - b)v_1$$

shows that $a > c > 0$ and $d > b > 0$. This implies $ad - bc > 1$, again contradicting (3.3). □

Exercise 3.4. This exercise will show that $a_1, \ldots, a_d$ from (3.1) satisfy $\sum_{i=1}^d a_i = 3d - 12$.

a. Verify the identity for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hirzebruch surfaces.

b. Prove that if the identity holds for Δ, then it holds for any stellar refinement of Δ.

c. Prove that the identity holds for all smooth complete toric surfaces.

The identity $\sum_{i=1}^d a_i = 3d - 12$ will have an nice application in §4.

Intersection Numbers. We begin with a brief discussion of the intersection number $(C \cdot D)$ of two irreducible curves on a smooth variety S of dimension 2. If $C \neq D$, the intersection $C \cap D$ is a finite set of points, and we define

$$(C \cdot D) = \sum_{p \in C \cap D} I_p(C, D), \tag{3.4}$$

where $I_p(C, D)$ is the *local intersection multiplicity* defined by

$$I_p(C, D) = \dim_{\mathbb{C}} \mathcal{O}_{S,p}/\langle f_p, g_p \rangle.$$

Here, $f_p = 0$ (resp. $g_p = 0$) is a local equation for C (resp. D) in a Zariski neighbhorhood of p.

Exercise 3.5. Let $X = X(\Delta)$ be a smooth complete toric surface. Let $\sigma \in \Delta(2)$ and let ρ_i, $i = 1, 2$ be the edges of σ. Then let $D_i = V(\rho_i) \simeq \mathbb{P}^1$ be the corresponding orbit closures in X. Prove that $(D_1 \cdot D_2) = 1$. Hint: Work in $U_\sigma \simeq \mathbb{C}^2$.

We next discuss the self-intersection $(C \cdot C)$. We can't define this using (3.4) since $C \cap C$ is infinite. This is a common problem in intersection theory—what do we do when an intersection has the wrong dimension? The standard method is to take a Chern class of an appropriate *excess intersection bundle*. For $(C \cdot C)$, the excess intersection bundle is the *normal bundle* of $C \subset S$, which is defined as follows. Using the structure sheaf $\mathcal{O}_C$ on C and the inclusion $i : C \subset S$, we get the sheaf $i_*\mathcal{O}_C$ on S defined by

$$i_*\mathcal{O}_C(U) = \mathcal{O}_C(U \cap C).$$

Then restriction of functions gives an exact sequence of sheaves

$$0 \to \mathcal{I}_C \to \mathcal{O}_S \to i_*\mathcal{O}_C \to 0.$$

We call $\mathcal{I}_C$ the *ideal sheaf* of C in S. In Lecture 9, we will see that $\mathcal{I}_C = \mathcal{O}_S(-C)$ since C is a Cartier divisor on S. The *conormal sheaf* $\mathcal{I}_C/\mathcal{I}_C^2$ has the natural structure of a sheaf of $\mathcal{O}_C$-modules, and we define the *normal sheaf* of C in S to be the dual sheaf

$$\mathcal{N}_{S/C} = \underline{Hom}_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C).$$

In general, this sheaf can be badly behaved, but here, $\mathcal{N}_{S/C}$ is an *inverible sheaf* (also called a *line bundle*). Since this is a line bundle on a curve, the Chern class $c_1(\mathcal{N}_{S/C})$ is an integer called the *degree* of $\mathcal{N}_{S/C}$, written $\deg(\mathcal{N}_{S/C})$. (Some of these terms will be defined in Lecture 9).

Then the *self-intersection* $(C \cdot C)$ is defined by

$$(C \cdot C) = \deg(\mathcal{N}_{S/C}). \tag{3.5}$$

We now apply this theory to smooth toric surfaces.

Self-Intersections on Smooth Toric Surfaces. Let $X = X(\Delta)$ be a smooth toric surface, and let $\sigma_1, \sigma_2 \in \Delta(2)$ be adjacent cones. If the minimal generators of σ_1 (resp. σ_2) are v_1, v_2 (resp. v_2, v_3), then Exercise 3.2 implies

$$v_1 + v_3 = av_2$$

for $a \in \mathbb{Z}$. Consider the edge $\rho = \sigma_1 \cap \sigma_2 = \mathrm{Cone}(v_2)$. In Lecture 9, we will prove that the normal bundle of the orbit closure $V(\rho) \simeq \mathbb{P}^1$ in X is $\mathcal{O}_{\mathbb{P}^1}(-a)$. We will also see that $\mathcal{O}_{\mathbb{P}^1}(-a)$ has degree $-a$. Then (3.5) will imply the following result.

Proposition 3.3. *$D = V(\rho)$ is a curve on X with self-intersection $(D \cdot D) = -a$.*

This proposition has some nice consequences. In the situation of the classification theorem, we have $X = X(\Delta)$ with minimal generators v_i satisfying

$$v_{i-1} + v_{i+1} = a_i v_i$$

by Exercise 3.2. The edge $\rho_i = \mathrm{Cone}(v_i)$ gives the orbit closure $D_i = V(\rho_i) \simeq \mathbb{P}^1$ on X, and then Proposition 3.3 implies that $(D_i \cdot D_i) = -a_i$ for all i. Furthermore, Proposition 3.3 justifies the claims made about self-intersections in the discussions following (2.1), (2.2), and (2.5).

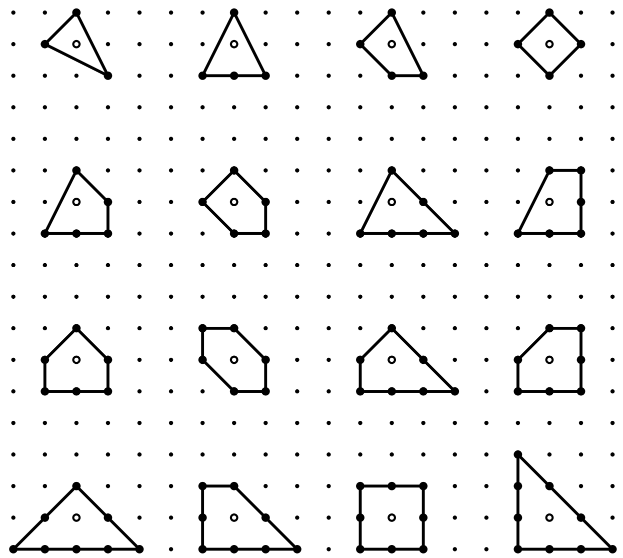
§4. Lattice Polygons and the Number 12

Our discussion will be based on the paper [6]. Many proofs will be omitted. One of our goals is to illustrate some of the many interesting things connected with toric varieties.

Reflexive Polygons. Let $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^2$ be a 2-dimensional polytope which is the convex hull of a set of points in N. We say that N is a *lattice polygon.*

Definition 4.1. *A lattice polygon $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^2$ is* **reflexive** *if there is exactly one lattice point in the interior of P.*

If P is a reflexive polygon and $\gamma \in \mathrm{Aut}_{\mathbb{Z}}(N) \simeq \mathrm{GL}(2,\mathbb{Z})$, then one easily sees that $\gamma(P)$ is also reflexive. We say that P and $\gamma(P)$ are *equivalent*. Up to equivalence (and allowing translation by elements of N), there are precisely 16 reflexive polygons in the plane:



The 16 equivalence classes of reflexive polygons (from [6])

If $P \subset N_{\mathbb{R}} \subset \mathbb{R}^2$ is a polygon with the origin in its interior, then its *dual* or *polar* is

$$P^\circ = \{u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq -1 \text{ for all } v \in P\}.$$

One can show that 0 is an interior point of P°, and one has the duality $P = (P^\circ)^\circ$ for polygons in the plane containing the origin in their interior.

In general, the dual of a lattice polygon need not be a lattice polygon.

Exercise 4.1. Let $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^2$ be the lattice polygon with vertices $\pm 2e_1, \pm 2e_2$, where e_1, e_2 form a basis of N. Prove that P° has vertices $\frac{1}{2}(\pm e_1^* \pm e_2^*)$ in $M_{\mathbb{R}}$.

We can characterize reflexive polygons in the plane as follows.

Proposition 4.2. *Let P be a lattice polygon P containing 0 in its interior. The P is reflexive if and only if P° is reflexive.*

Exercise 4.2. In the illustration on the previous, find the dual of each reflexive polygon. (When computing the dual, assume that the origin is the unique interior lattice point.) Note that some of the polygons are self-dual.

We should also mention that in higher dimensions, the definition of reflexive requires more than just a unique interior lattice point (as in Definition 4.1). This will be covered in the lectures on mirror symmetry given next week.

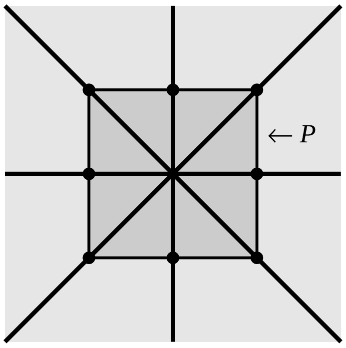
Lattice Points of Reflexive Polygons. Given a reflexive polygon P, we let $\ell(P)$ denote the number of lattice points on the boundary of P.

Theorem 4.3. *Let P be a reflexive polygon containing the origin. Then*

$$\ell(P) + \ell(P^\circ) = 12.$$

The paper [6] gives four proofs of this result. The simplest uses the illustration on the previous page together with Exercise 4.2. There is also a proof using modular forms. For us, the proof of interest uses toric varieties and the Noether formula. In fact, we will give two "toric" proofs, only one of which uses the Noether formula.

Given a reflexive polygon P containing the origin, we get a fan Δ_P by using the lattice points on the boundary of P as cone generators. Here is an example of what this looks like:



Note that Δ_P has $\ell(P)$ edges and $\ell(P)$ 2-dimensional cones. This fan is obviously complete, and the following exercise shows that Δ_P is also regular.

Exercise 4.3. Let $v_1, v_2 \in N \simeq \mathbb{R}^2$ be the minimal generators of a 2-dimensional strongly convex rational polyhedral cone σ. Prove that σ is regular if and only if $\mathrm{Conv}(0, v_1, v_2) \cap N = \{0, v_1, v_2\}$. Hint: Let e_1, e_2 be a normalized basis for σ and let m, k be the parameters of σ. Show that $m > 1$ implies $e_1 \in \mathrm{Conv}(0, v_1, v_2)$.

In this construction, it is important to use *all* lattice points on the boundary. For example, if we only used the vertices of P in the above example, the resulting fan would not be regular.

For later purposes, we will also need to discuss the *area* of a polygon in $N_{\mathbb{R}} \simeq \mathbb{R}^2$. Let e_1, e_2 be a basis of N. Then we will measure area so that the square with vertices $0, e_1, e_2, e_1 + e_2$ has area 1. (Since elements of $\mathrm{GL}(2, \mathbb{Z})$ have determinant ± 1, this definition is independent of the chosen basis.) We can then relate area to regularity as follows.

Exercise 4.4. Let σ and v_1, v_2 be as in Exercise 4.3. Prove that σ is regular if and only if $\mathrm{Conv}(0, v_1, v_2)$ has area $\frac{1}{2}$.

Exercise 4.5. Let P be a reflexive polygon. Prove that $\mathrm{Area}(P) = \frac{1}{2}\ell(P)$.

Self-Intersections on Smooth Toric Surfaces. Before we present the toric proofs of Theorem 4.3, we first discuss another way to compute self-intersections. Suppose that $X = X(\Delta)$ be a smooth complete toric surface, and let $\rho_1, \dots, \rho_d$ denote its edges, and let v_i be the minimal generator of ρ_i. This gives orbit closures $D_i = V(\rho_i) \simeq \mathbb{P}^1$ in X.

Now consider a formal sum $D = \sum_{i=1}^d b_i D_i$, where $b_i \in \mathbb{Z}$. In the terminology of Lecture 9, D is a T*-invariant divisor* on X. From D, we get the polygon $Q \subset M_{\mathbb{R}}$ defined by

$$Q = \{u \in M \mid \langle u, v_i \rangle \geq -b_i \text{ for all } i\}. \tag{4.1}$$

Then later in the week, you will see that if $b_i \geq 0$ for all i, then

$$(D \cdot D) = 2\,\mathrm{Area}(Q). \tag{4.2}$$

(See also page 111 of [2].) This formula can be used to prove Theorem 4.3 as follows.

First Proof of Theorem 4.3. Given a reflexive polytope P containing the origin, we get the complete regular fan Δ_P described above. Then $X = X(\Delta_P)$ is a smooth complete toric surface. Now consider the divisor $-K = \sum_{i=1} D_i$ (the reason for calling this $-K$ will become clear in the second proof). For this divisor, the polytope (4.1) is

$$Q = \{u \in M \mid \langle u, v_i \rangle \geq -1 \text{ for all } i\}.$$

Since P is the convex hull of the v_i, it follows that

$$Q = \{u \in M \mid \langle u, v \rangle \geq -1 \text{ for all } v \in P\} = P^\circ.$$

Hence the polytope associated to the divisor $-K$ is the dual reflexive polygon P°. This implies

$$(-K \cdot -K) = 2\,\mathrm{Area}(P^\circ) = \ell(P^\circ), \tag{4.3}$$

where the first equality is (4.2) and the second is by Exercise 4.5 (since P° is reflexive by Proposition 4.2). However, we can also compute $(-K \cdot -K)$ as follows:

$$(-K \cdot -K) = \Big(\sum_{i=1}^d D_i \cdot \sum_{i=1}^d D_i\Big) = \sum_{i=1}^d (D_i \cdot D_i) + 2\sum_{i<j}(D_i \cdot D_j).$$

We saw at the end of §3 that $(D_i \cdot D_i) = -a_i$, and we also know that for $i < j$,

$$(D_i \cdot D_j) = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the above computation simplifies to

$$(-K \cdot -K) = -\sum_{i=1}^d a_i + 2d = -(3d - 12) + 2d = 12 - d,$$

where the second equality uses Exercise 3.5. Here, d is the number of edges in Δ_P, which is $\ell(P)$ by the construction of Δ_P. Comparing this with (4.3), we obtain

$$12 - \ell(P) = (-K \cdot -K) = \ell(P^\circ).$$

This completes the first proof. □

Noether's Formula. Before we can give the second proof, we need to explain Noether's formula. Let X be a smooth complete surface. Then Noether's formula states that

$$12\chi(\mathcal{O}_X) = (K \cdot K) + \chi(X). \tag{4.4}$$

Here are what the components of this formula mean:

- $\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) + \dim H^2(X, \mathcal{O}_X)$. This is *holomorphic Euler characteristic*. Sheaf cohomology will be discussed later in the week.
- $\chi(X) = \sum_{q=0}^{4} (-1)^q b_q(X)$, where the *qth Betti number* $b_q(X)$ is the rank of the singular homology group $H_q(X)$. This is the usual *Euler characteristic* from algebraic topology.
- K is a *canonical divisor*, which means that the sheaf $\mathcal{O}_X(K)$ is isomorphic to the sheaf Ω_X^2 of algebraic 2-forms on X. Lecture 9 will explain how any Cartier divisor D gives an invertible sheaf $\mathcal{O}_X(D)$.

The Noether formula is discussed in detail in [3, Chapter 4] and [4, A.4]. We should also mention that Noether's formula can be stated using the tangent bundle $\mathcal{T}_X$ of X. Namely, one can show that $(K \cdot K) = \int_X c_1(\mathcal{T}_X) \cup c_1(\mathcal{T}_X)$ and $\chi(\mathcal{O}_X) = \int_X c_2(\mathcal{T}_X)$, where $c_1(\mathcal{T}_X)$ and $c_2(\mathcal{T}_X)$ are the *Chern classes* of the vector bundle $\mathcal{T}_X$.

Now let $X = X(\Delta)$ be a smooth complete toric surface. We will see in later lectures that the following holds:

- $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. Thus $\chi(\mathcal{O}_X) = 1$.
- The Euler characteristic of X is d, the number of cones in $\Delta(2)$.
- The canonical divisor of X is $K = -\sum_{i=1}^{d} D_i$.

Thus, Noether's formula (4.4) for a smooth complete toric surface can be written

$$12 = (-K \cdot -K) + d, \tag{4.5}$$

where d is the cardinality of $\Delta(2)$ and $-K = \sum_{i=1}^{d} D_i$.

Second Proof of Theorem 4.3. Let $X = X(\Delta_P)$ be the smooth complete toric surface coming from the reflexive polytope P. Then $(-K \cdot -K) = \ell(P^\circ)$ by (4.3), and $d = \ell(P)$ follows from the construction of Δ_P. Then Theorem 4.3 is an immediate consequence of (4.5). $\square$

Exercise 4.5. In the first proof of Theorem 4.3, we showed that $(-K \cdot -K) = 2d - \sum_{i=1}^{d} a_i$. Use this to prove that (4.4) is equivalent to the identity $\sum_{i=1}^{d} a_i = 3d - 12$ of Exercise 3.5.

In one sense, it is overkill to use Noether's formula to prove Theorem 4.3. Our reasons for giving the proof were to introduce some ideas you will see later in the week and to illustrate how fundamental invariants of toric varieties (such as $(K \cdot K)$ and $\chi(X)$) can be computed from combinatorial invariants of the fan. We also wanted to give a simple example of the deep link between polytopes and toric varieties.

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