Gabby and Eliza,

The presentation on Lagrange Multipliers as an application of Gröbner basis methods, with economics applications was OK, but it felt as though you hadn't practiced giving the talk enough (or hadn't worked much on it between the run-through on Thursday and when you presented on Monday) to feel really comfortable explaining the technical material. There were a number of small slips and somewhat unclear statements. The basic ideas were clear enough and I think everyone understood where you were coming from and how the mathematics we learned in the seminar could be applied.

The paper is better than the talk, although I have a lot of comments about various points. The first and general comment is that there are a lot of typos and small language issues. You need to proofread a paper for a math course just the way you do for other courses. Your capitalization is really inconsistent, for one small thing. All instances of Lagrange and Gröbner should be capitalized because they are proper names. The second general comment is that you need to be more careful about giving credit for graphs and other images you take from sources. They should be attributed with footnotes just the way direct quotations and paraphrases of information from your sources are footnoted. I'm not taking off any credit for this, but you should be aware that using figures without attribution is often seen as a form of plagiarism.

Specific Comments

1. I don't think it's correct to say that Cobb-Douglas production functions are "specific to econometrics." I saw them in the Introduction to Economics course I took as an undergraduate. It's probably better to say they are a widely-used class of model functions because they are relatively simple. You address some of the ways they might be unrealistic at the end of the paper, and that is good.
2. It's not exactly clear how this diagram relates to the situation where you have two distinct inputs and the graph would be a surface in 3-dimensional space (if you included a coordinate axis for the output). How do increasing/constant/decreasing returns connect with the exponents in a Cobb-Douglas function?
3. Economists probably do always make the assumption that increasing X and Y simultaneously takes you to an indifference curve with a higher value of utility. However, from a more philosophical viewpoint, it's probably worthwhile to ask whether having more of everything increases utility or well-being for people in real life. Are really rich people always happier than those with less money and possessions?
4. There's no theoretical reason why a utility function has to have the form $U(x, y) = xy$. This is just a convenient example.
5. What is a "tautochrone particle?" (I think you are referring to something called the tautochrone problem which was a famous question that stimulated the mathematical field called the *calculus of variations* (and that was the context for Lagrange's invention of the multiplier method). A *tautochrone* is a descending curve in a vertical plane with the property that if you have a bead or particle sliding along the curve, acted on by gravity, then it takes the same amount of time for the particle to slide down to the end no matter what point it is released from. (The name *tautochrone* comes from Greek

words meaning "same time" in fact.) The problem was to find the equation of such a curve in terms of known functions.

6. This is a big overstatement. People were using and applying Cobb-Douglas functions in constrained optimization long before Gröbner bases were even invented(!)
7. What does it mean to say this was "only a theory?" Isn't everthing in mathematics a theory? (I mean that every statement in mathematics has the form "if some hypotheses are true, then some conclusions follow." Mathematicians never claim that they have absolute truth or a description of the way the universe really works!)
8. I don't know what you mean by "finessed" here. Macaulay wasn't the only one to work on Elimination Theory. This was a central area of mathematics in the 19th century and lots of people did lots of different things related to eliminating variables from systems of equations. One of the most important ideas we didn't discuss in the course uses determinants called *resultants* to do elimination. This is discussed in Chapter 3 of IVA.
9. Saying "optimal number of attacks" without more explanation sounds really weird and cold-hearted. The use of the words "utility" or "well-being" strikes me the same way. Wouldn't the optimal number of attacks be *zero* from the point of view of the *victims'* well-being? It would be good to say that this whole discussion is from the point of view of the terrorists, and the issue is how they can use the resources at their disposal to do the most damage. The whole point is to understand how that damage would change if the cost of hitting a political target goes up. The number of civilian attacks would go up, while the total damage done would decrease (that's the shift to the lower "indifference curve" for the terrorists).
10. This is not correct (it wasn't correct in the talk slides either). The idea is that if the ideal is $I = \{0\}$, then to say you have a Gröbner basis too in that case, you would take the Gröbner basis to be $G = \emptyset$.
11. You didn't really finish this example! What are the corresponding values of y, x? Which point (x, y, z) maximizes $f(x, y, z) = x^3yz$ on the constraint set?
12. This is not clear at all. I think what you are trying to say is that some industries (like agriculture) involve many more inputs than just two as in the basic Cobb-Douglas function, so that is not a realistic model. Is that what you meant?

Final Project Presentation: 82 (B-)

Final Project Paper: 85 (B)