MATH 392 – Seminar in Computational Commutative Algebra
Midterm Exam
March 22, 2019

Directions

Do all work in the blue exam book. Do not place anything you want considered for credit on this sheet. There are 100 possible points, distributed as indicated in the questions.

I.

A) (20) Prove that every ideal I in the polynomial ring $k[x]$ (one variable) is principal (that is, $I = \langle g(x) \rangle$ for some single polynomial).

B) (10) Find $g(x)$ as in part A for the ideal $I = \langle x^2 + 7x + 10, x^3 + x^2 + 4 \rangle$ in $\mathbf{Q}[x]$.

II.

A) (15) Define: G is a Gröbner basis for an ideal $I \subseteq k[x_1, \ldots, x_n]$ with respect to a monomial order $>$.

B) (10) Assuming the statement of Dickson's Lemma, prove that Gröbner bases exist for every ideal I in $k[x_1, \ldots, x_n]$ and with respect to every monomial order.

C) (5) What else do you need to know in order for the result from part B to give a proof of the Hilbert Basis Theorem? (You don't need to give the proof, just say what else must be proved.)

III.

A) (20) State and prove the Elimination Theorem.

B) (10) A certain ideal $J \subset \mathbf{Q}[x, y, z]$ has a Gröbner basis

$$B = \{x^3 - 3x^2 + 2x, x^2y - xy, y^2 - y, z - xy\}$$

with respect to the lexicographic order with $z > y > x$. What are bases for the elimination ideals

$$J_1 = J \cap \mathbf{Q}[y, x] \quad \text{and} \quad J_2 = J \cap \mathbf{Q}[x]?$$

C) (10) Use the information in part B to determine all of the points in $V(J)$.