SIMU 1998–Gröbner Bases Seminar
Week 3 Computer Laboratory Exercises

Background and Goals

This week, we will work through examples of Buchberger's Algorithm for Gröbner bases using new functions from the `Groebner` package, and then introduce the full implementation of that algorithm in the `gbasis` command.

Lab Work

We have now seen the definition of a Gröbner basis for an ideal $I \subset k[x_1, \ldots, x_n]$ (with respect to a given monomial order) – it is a collection of polynomials $g_1, \ldots, g_t \in I$ such that the monomial ideals $\langle LT(I) \rangle$ and $\langle LT(g_1), \ldots, LT(g_t) \rangle$ are equal (or equivalently, such that the leading term of every $f \in I$ is divisible by $LT(g_i)$ for some $1 \leq i \leq t$). Specifying the monomial order is necessary, of course, to identify the leading terms. A Gröbner basis for I with respect to one monomial order will rarely be a Gröbner basis for I with respect to a "randomly chosen" second monomial order too.

Today, we want to consider whether or not given sets of generators for an ideal are Gröbner bases and do some examples related to Buchberger's Criterion:

$G = \{g_1, \ldots, g_t\} \subset I$ is a Gröbner basis for I if and only if the S-polynomial remainder $\overline{S(g_i, g_j)}^G$ is equal to zero for all $1 \leq i < j \leq t$.

We will also consider the Buchberger Algorithm.

The building blocks we will need are the computation of S-polynomials and the remainder computation we used before with `normalf`. The command for S-polynomial calculation in the `Groebner` package is called `spoly` (note the lower-case s). The format is

```
spoly(p,q,monomialorder);
```

where p and q are the polynomials, and `monomialorder` is specified as before. One important thing to note here: The S-polynomial computed by Maple will only agree with the results of the formula in "IVA" up to a non-zero constant multiple. This is because Maple's command is set up *not to introduce denominators* when the coefficients of the original polynomial are integers. In other words, the result computed by the command `spoly(p,q,monomialorder);` is

$$\mathrm{lcm}(LT(p), LT(q)) \cdot \left(\frac{1}{LT(p)} \cdot p - \frac{1}{LT(q)} \cdot q \right)$$

(where the lcm includes the lcm of the integer coefficients as well). For example, the Maple command

```
spoly(3*x*y - x,4*y^2 - 1,plex(x,y));
```

produces output

$$-4xy+3x$$

which is different than what you would get using the formula in "IVA":

$$S(3xy-x,4y^2-1)=\frac{y}{3}(3xy-x)-\frac{x}{4}(4y^2-1)=\frac{-1}{3}xy+\frac{1}{4}x$$

The two results differ by a factor of 12. This does not affect the usability of Maple's result in any way though (it is another *implementation choice*, made to avoid potential "blow-ups" in the amount of space needed to store the coefficients in complicated polynomials).

Exercise 1

a) Verify by Buchberger's Criterion that $G=[x+z^3,y-z^7]$ is a Gröbner basis for the ideal it generates with respect to the *lex* order, $x>y>z$.
b) Divide $f=xy$ by G and also by $G'=[y-z^7,x+z^3]$ (reversing the order of the divisors) using the `DivQuots` procedure from last week. Are the remainders the same? Are the quotients the same?

Exercise 2

Do problem 9, parts a,c from Chapter 2, §6 of "IVA".

Buchberger's Algorithm

The Buchberger Criterion we quoted above leads directly to the rudimentary algorithm for computing a Gröbner basis of an ideal with respect to a given monomial order discussed in Chapter 2, §7 in "IVA". The idea is to compute S-polynomials of all pairs of polynomials from the list we start with, take the remainders of those S-polynomials on division by the current list of polynomials, adjoin any non-zero remainders to the list, and repeat until no more changes occur (that is, until all the remainders are zero). To get a feeling for how this actually works before we start letting Maple do *all the work* for us, we want to do a few examples of this "by hand" as in Example 1 from §7 of Chapter 2.

Exercise 3

Verify that all the necessary S-polynomial remainders are in fact zero in the final list of polynomials from Example 1 in §7 of Chapter 2.

Exercise 4

Do problem 2, parts a and b from that section.

Exercise 5

Develop a Maple procedure to perform the "basic" Buchberger algorithm presented in Theorem 2 of Chapter 2, §7. Use the `ExtEuc` and `DivQuots` procedures as models for the Maple syntax. You can also consult the Maple "programming primer" in the file `f:/users/simu/Little/primer.mws` as a reference.

"The Big Guns": The `gbasis` *Command*

In realistic examples, doing Buchberger's Algorithm "by hand", or even by a procedure implementing the rudimentary form of the algorithm from Theorem 2 of §7 of Chapter 2, would be hopelessly slow. Fortunately, numerous improvements have been made (see the optional §9 from Chapter 2 of "IVA" if you are interested in learning more about some of them) and implemented in the `gbasis` command of Maple (and similar packages for other computer algebra systems such as *Mathematica*, Axiom, REDUCE, etc.)

The `gbasis` *Command*

The "workhorse" of the `Groebner` package is the `gbasis` command, which computes (up to constant multiples) the unique *reduced Gröbner basis* of the ideal generated by a given collection of polynomials with respect to a given monomial order. Reduced Gröbner bases $G = \{g_1, \ldots, g_t\}$ are ones in which the leading terms form a minimal set of generators for $\langle LT(I) \rangle$, and in each g_i, there are no terms divisible by the $LT(g_j)$ for $j \neq i$. In other words, the g_i are "inter-reduced". The format is

```
gbasis(Plist,monomialorder);
```

where *Plist* is a list of polynomials, and monomialorder is defined as before. For example,

```
gbasis([x*y + y + 1,x*y^2 - 2*x + 1],plex(x,y));
```

produces the output

$$[-3y - 2 + y^3 + y^2, 2x - 1 + y^2 + y]$$

which is (up to constant multiple of the second polynomial) the reduced *lex* Gröbner basis for the ideal generated by the original two polynomials ($x > y$).

(One comment: The polynomials in *Plist* can contain *other variables* besides the ones set up in the definition of the polynomial ring, and/or the definition of the monomial order. If that is true, then those additional variables are treated as symbolic parameters in the coefficients. For example

```
gbasis([x*y + y + a,x*y^2 - 2*x + 1],plex(x,y));
```

produces the output:

$$[y^3 + y^2 a - 3y - 2a, 2x + y^2 + ay - 1]$$

This raises an important and interesting question – can we always substitute constant values for a into the general form of the Gröbner basis to obtain the basis for the corresponding ideal with constant coefficients? It worked here, but for example, suppose you had a symbolic parameter appearing in a leading term, and you substituted the value zero. What would happen then??)

A "Sampler" of Applications

Exercise 6 – Ideal Membership

Do problem 1 from Chapter 2, §8 of "IVA".

Exercise 7 – Solving Systems of Equations

a) Find the points in $\mathbf{V}(x^2y - z^3, 2xy - 4z - 1, z - y^2)$ in $\mathbf{R}^3$. Suggestions: Compute a *lex* Gröbner basis (say with $x > y > z$), and call the Gröbner basis H. Identify a polynomial containing only z in your output; it should be the first polynomial in the list, or `H[1]`. Solve the equation obtained by setting that polynomial equal to zero using the command `fsolve(H[1],z,complex);` to find approximations to all the complex roots. The `fsolve` command uses an approximate *numerical method* to solve equations. Then substitute each real root back into the other equations (use `subs` – see the online help) and solve them for the other variables.
b) Do problem 4 from Chapter 2, §8 of "IVA". (The system here has the first three polynomials the same as the three polynomials in part a, plus an additional fourth polynomial.) What does the Gröbner basis computation tell you here?

Exercise 8 – Implicitization of Parametric Curves and Surfaces

a) Problem 7, Chapter 2, §8 of "IVA".
b) Problem 8, Chapter 2, §8 of "IVA".
c) Problem 9, Chapter 2, §8 of "IVA".
d) Generate a composite graphics image showing the parametric curve from problem 9, and an implicit plot of the surface from problem 8. How does this relate to your answer to part c of Problem 9?

"Extra Credit": Exercise 9 – Identities

Do problem 11, Chapter 2, §8 of "IVA".