Mathematics 375 – Probability Theory
Solutions for Midterm Exam 2 – November 17, 2011

I. Birth weights of babies born in the US are normally distributed with mean $\mu = 3315$ grams and standard deviation $\sigma = 575$ grams.

A) (5) Find the (approximate) probability that a baby will have birth weight more than 3700 grams.

Solution: Let Y = birth weight of a randomly selected baby, so by the given information, $Y \sim Normal(3315, 575^2)$. Standardizing, we have

$$P(Y > 3700) = P\left(\frac{Y - 3315}{575} > \frac{3700 - 3315}{575}\right) \doteq P(Z > .67) \doteq .2514$$

(since $\frac{Y-3315}{575} = Z \sim Normal(0,1)$).

B) (10) If 20 newborn babies are selected randomly, how many of them would be expected to have weights between 2740 and 3890 grams?

Solution: We have $2740 = 3315 - 575 = \mu - \sigma$ and $3890 = \mu + \sigma$. The probability that a single baby has a weight in this range is

$$P(2740 < Y < 3890) = P(-1 < Z < 1) = 1 - 2P(Z > 1) = 1 - 2(.1587) = .6826$$

(Note that this interval is where the 68% in the Empirical Rule comes from.) If X is the number of babies in the 20 that have weights in this range, then technically X would have a *hypergeometric* distribution, since the selection *would not be done with replacement*. However, we don't know the size of the entire population of babies in question. In any case that would be much larger than 20. So the *binomial* formulas can be used to estimate this. Say $X \sim Binomial(20, .6826)$. Then

$$E(X) = n \cdot p = 20 \cdot 0.6826 = 13.652.$$

So we expect this many babies to have weights in this range.

C) (10) If we knew the mean and standard deviation, but we did not know the precise distribution of the birth weights, what could we say in general about the probability that a randomly selected baby has weight differing from 3315 grams by at least 862.5 grams?

Solution: By Tchebysheff's Theorem, since $862.5 = 1.5 \cdot 575$

$$P(|Y - \mu| \geq 1.5\sigma) \leq \frac{1}{(1.5)^2} = .4444.$$

II. (15) The number of defects Y in a 1-foot segment of a magnetic tape is a Poisson random variable with mean 3. What is the probability that a 1-foot segment contains more than 4 defects?

Solution: This is

$$P(Y > 4) = \sum_{y=5}^{\infty} \frac{3^y e^{-3}}{y!} = 1 - \sum_{y=0}^{4} \frac{3^y e^{-3}}{y!} \doteq .1847.$$

III. (10) (From an actuarial "P Exam" problem) The warranty on a machine specifies that it will be replaced at failure or at age 4, whichever occurs first. The machine's age at failure, X, is a random variable with probability density function

$$f(x) = \begin{cases} 1/5 & \text{if } 0 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the age of the machine *at replacement*. What is $E(Y)$?

Solution: Note that X has a uniform distribution on the interval $[0, 5]$. The age of the machine at replacement is given by this function of X:

$$Y = \begin{cases} X & \text{if } 0 \le X < 4 \\ 4 & \text{if } 4 \le X \le 5 \\ 0 & \text{otherwise} \end{cases}$$

So

$$E(Y) = \int_0^4 x \cdot \frac{1}{5}\ dx + \int_4^5 4 \cdot \frac{1}{5}\ dx = \frac{16}{10} + \frac{4}{5} = \frac{12}{5}.$$

IV.

A) (10) Let

$$g(x) = \begin{cases} \frac{1}{16} x^2 e^{x/2} & \text{if } x < 0 \\ 0 & \text{otherwise (that is, if } x \ge 0). \end{cases}$$

Show that g is a legal pdf for a random variable X.

Solution: $g(x) \ge 0$ for all x by inspection. Moreover

$$\int_{-\infty}^{\infty} g(x)\ dx = \int_{-\infty}^{0} \frac{1}{16} x^2 e^{x/2}\ dx$$

Let $y = -x/2$. Then this integral becomes

$$\frac{1}{2} \int_0^{\infty} y^2 e^{-y}\ dy = \frac{1}{2}\Gamma(3) = 1.$$

Therefore all the requirements for a pdf are satisfied.

B) (10) Determine the moment-generating function of X. On which interval of t values is your formula valid?

Solution: The mgf is

$$E(e^{tX}) = \int_{-\infty}^{0} e^{tx} \cdot \frac{1}{16} x^2 e^{x/2}\, dx$$
$$= \int_{-\infty}^{0} \frac{1}{16} x^2 e^{(1+2t)x/2}\, dx$$

Let $y = -(1+2t)x/2$ or $x = \frac{-2y}{1+2t}$. Making the substitution, we get (provided $1+2t > 0$),

$$= \int_{0}^{\infty} \frac{1}{2(1+2t)^3} y^2 e^{-y}\, dy$$
$$= \frac{1}{2(1+2t)^3} \Gamma(3)$$
$$= \frac{1}{(1+2t)^3}$$

In order for this to be valid, the improper integral must converge, and for that $1+2t > 0$, or $t > -1/2$.

V. Let Y_1, Y_2 be jointly continuous random variables with joint density

$$f(y_1, y_2) = \begin{cases} cy_1 & \text{if } 0 \le y_1 \le 1 \text{ and } y_1^2 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

A) (15) What is $P(Y_1 \le Y_2)$?

Solution: The region where the density is nonzero is the region with $0 \le y_1 \le 1$, above the parabola $y_2 = y_1^2$, and below the line $y_2 = 1$. The value of c is determined as follows. We must have

$$1 = \int_0^1 \int_{y_1^2}^1 cy_1\, dy_2\, dy_1$$
$$= c \int_0^1 y_1 - y_1^3\, dy_1$$
$$= c\left(\frac{1}{2} - \frac{1}{4}\right)$$

So $c = 4$.
The condition $Y_1 \le Y_2$ then defines the triangle with vertices at $(0,0), (1,1), (0,1)$.

$$P(Y_1 \le Y_2) = \int_0^1 \int_{y_1}^1 4y_1\, dy_2\, dy_1$$
$$= \int_0^1 4y_1 - 4y_1^2\, dy_1$$
$$= 2y_1^2 - \frac{4}{3} y_1^3 \Big|_0^1$$
$$= \frac{2}{3}.$$

B) (15) Find the marginal density $f_1(y_1)$ and use it to compute $V(Y_1)$.

Solution: The marginal density is

$$f_1(y_1) = \begin{cases} \int_{y_1^2}^1 4y_1 \; dy_2 = 4y_1(1-y_1^2) & \text{if } 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E(Y_1) = \int_0^1 4y_1^2 - 4y_1^4 \; dy_1 = \frac{4}{3} - \frac{4}{5} = \frac{8}{15}.$$

Similarly

$$E(Y_1^2) = \int_0^1 4y_1^3 - 4y_1^5 \; dy_1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

So

$$V(Y_1) = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}.$$

Extra Credit. (10) What is $E(X)$ for the random variable X from question IV?

Solution: The answer is $E(X) = -6$. This can be computed either directly by evaluating $E(X) = \int_{-\infty}^0 x \cdot g(x) \; dx$, or more cleverly by using the mgf from the answer to IV B):

$$E(X) = \frac{d}{dt} \frac{1}{(1+2t)^3}\bigg|_{t=0} = \frac{(-3)(2)}{(1+2\cdot 0)^4} = -6.$$

(Note: the density $g(x)$ has the form of a gamma density with $\alpha = 3$ and $\beta = 2$, but reflected about $x = 0$. So the expected value should be the negative of the expected value of the corresponding standard gamma density: $\alpha\beta = 6$.)