Mathematics 376 – Mathematical Statistics
Solutions for Midterm Exam 2
April 26, 2012

I. Let $Y_1, \ldots, Y_n$ be a random sample from a population with density function

$$f(y|\theta) = \begin{cases} \frac{4y^3}{\theta^4} & \text{if } 0 \le y \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

A) (10) Determine the method of moments estimator for $\theta$.

*Solution:* Computing $E(Y)$ from this density, we find:

$$E(Y) = \int_0^\theta y \cdot \frac{4y^3}{\theta^4}\, dy = \frac{4\theta}{5}.$$

For the method of moments, we set this equal to $\overline{Y}$, the first sample moment, and solve for $\theta$:

$$\widehat{\theta}_{MOM} = \frac{5\overline{Y}}{4}.$$

B) (10) Determine the maximum likelihood estimator for $\theta$.
*Solution:* The likelihood function in this case is

$$L(y_1, \ldots, y_n \mid \theta) = \frac{4^n (\prod_{i=1}^n y_i)^3}{\theta^{4n}} \tag{1}$$

As a function of $\theta$, neither this nor

$$\ln(L) = n\ln(4) + 3\sum_{i=1}^n \ln(y_i) - 4n\ln(\theta)$$

has a critical point. Both are strictly *decreasing* as functions of $\theta$. Therefore, we want to choose $\theta$ as small as possible, consistent with the sample values, in order to maximize $L$. This occurs when $\theta = Y_{(n)}$, the sample maximum. Therefore

$$\widehat{\theta}_{ML} = Y_{(n)}.$$

C) (5) Is $U = \sum_i \ln(Y_i)$ a sufficient statistic for $\theta$? Why or why not?
*Solution:* The answer is *yes* by the factorization criterion. From (1) above we have

$$L(y_1, \ldots, y_n \mid \theta) = \frac{4^n e^{3U}}{\theta^{4n}} = g(U, \theta)$$

We can take the other factor $h(y_1, \ldots, y_n) = 1$.

II. Assume that the hardness of iron bars produced by one plant is normally distributed with mean $\mu$ and variance $\sigma^2$. A random sample of $n = 7$ bars from the output of the plant is measured to determine whether the average hardness meets the desired figure $\mu_0 = 172$.

A) (20) The following measurements:

$$167, 174, 179, 164, 163, 160, 168$$

were obtained. Test $H_0 : \mu = 172$ versus the alternative $H_a : \mu \neq 172$ using an appropriate test with $\alpha = .01$.

*Solution:* Since $n < 30$, we must use a $t$-test here. We have $\overline{Y} \doteq 167.86$ and $S^2 \doteq 43.81$. We have

$$t = \frac{167.82 - 172}{\sqrt{\frac{43.81}{7}}} \doteq -1.66$$

The rejection region for the test is determined from the $t$-table. The form of the alternative hypothesis indicates we want

$$RR = \{t \mid |t| > t_{.005}\} = \{t \mid |t| > 3.707\}$$

(for 6 df). There is not sufficient evidence to reject $H_0$ here.

B) (10) How is your answer from A related to the 99% confidence interval for $\mu$ that we learned to compute in the first section of the course?

*Solution:* The 99% confidence interval would be

$$167.82 \pm 3.707\sqrt{\frac{43.81}{7}} = 167.82 \pm 9.27 = (158.58, 177.13).$$

The value $\mu_0 = 172$ is one of the numbers in this interval, so it is a "believable" value for $\mu$ based on the data. Alternatively, the rejection region for the test is the *complement* of the interval

$$172 \pm 3.707\sqrt{\frac{43.81}{7}} = 172 \pm 9.27$$

Since the sample $\overline{Y}$ is in this interval we do not reject $H_0$.

III. In this problem, let $S$ represent the proportion of women who agree that men are basically selfish and egotistical.

A) (15) Suppose that surveys done 20 years ago showed that $S$ was 0.40 at that time, but a new survey finds that $Y = 1260$ out of a random sample of $n = 3000$ women said that men are basically selfish and egotistical. Is there sufficient evidence to say the proportion $S$ has increased? Give the null and alternative hypotheses for a suitable test, carry out the analysis, estimate the attained significance level ($p$-value) and state your conclusion succinctly and clearly.

*Solution:* The wording of the question is supposed to suggest using $H_0 : S = .4$ and $H_a : S > .4$. Since $n = 3000$ is large, we use a $Z$-test:

$$Z = \frac{\frac{1260}{3000} - .4}{\sqrt{\frac{(.4)(.6)}{3000}}} = 2.236.$$

For the upper tail test, we would reject $H_0$ for any $\alpha$ with $z_\alpha \leq 2.236$. From the standard normal table, this says the $p$-value is about .0125. There is relatively strong evidence to reject $H_0$.

B) (15) Suppose that the current value of $S$ is actually $S = .43$ and the test in part A was carried out to yield $\alpha = 0.05$. How large would $n$ have to be in a sample like the one from part A to ensure that $\beta$ (the probability of a Type II error) is .1 or less?

*Solution:* In terms of $\frac{Y}{n}$, the rejection region for the test with $\alpha = 0.05$ (and a general sample size $n$) is

$$\left\{ \frac{Y}{n} > .4 + 1.645\sqrt{\frac{(.4)(.6)}{n}} \right\}$$

Recall that $\beta$ is the chance of *not rejecting* $H_0$ when $H_0$ is false. If $S = .43$, we start from

$$P\left(\frac{Y}{n} \leq .4 + 1.645\sqrt{\frac{(.4)(.6)}{n}}\right)$$

and "restandardize" using $S = .43$:

$$\begin{aligned}
\beta &= P\left(\frac{Y}{n} \leq .4 + 1.645\sqrt{\frac{(.4)(.6)}{n}}\right) \\
&= P\left(\frac{\frac{Y}{n} - .43}{\sqrt{\frac{(.43)(.57)}{n}}} \leq \frac{-.03 + 1.645\sqrt{\frac{(.4)(.6)}{n}}}{\sqrt{\frac{(.43)(.57)}{n}}}\right) \\
&= P\left(Z \leq \frac{-.03 + 1.645\sqrt{\frac{(.4)(.6)}{n}}}{\sqrt{\frac{(.43)(.57)}{n}}}\right)
\end{aligned}$$

To get this to be .1, we want

$$\frac{-.03 + 1.645\sqrt{\frac{(.4)(.6)}{n}}}{\sqrt{\frac{(.43)(.57)}{n}}} \doteq -1.28$$

so

$$\sqrt{n} = \frac{1.645\sqrt{(.4)(.6)} + 1.28\sqrt{(.43)(.57)}}{.03} \doteq 47.98$$

so $n \geq 2303$ will do. (In other words, the test with $n = 3000$ will already have $\beta$ this small(!).)

IV. (15) Let $(x_i, y_i, z_i)$, $i = 1, \ldots, n$ be a collection of data points. Using the matrix formulation, derive the normal equations for the least squares estimators for the coefficients $\beta_0, \beta_1, \beta_2$ in the model $Z = \beta_0 + \beta_1 x + \beta_2 y + \varepsilon$ fitting the data.

*Solution:* The normal equations are $X^t X\beta = X^t Z$ where

$$X = \begin{pmatrix} 1 & x_1 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & x_n & y_n \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

Hence the normal equations can be written in matrix form as

$$\begin{pmatrix} n & \sum_i x_i & \sum_i y_i \\ \sum_i x_i & \sum_i x_i^2 & \sum_i x_i y_i \\ \sum_i y_i & \sum_i x_i y_i & \sum_i y_i^2 \end{pmatrix} \begin{pmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \\ \widehat{\beta_2} \end{pmatrix} = \begin{pmatrix} \sum_i z_i \\ \sum_i x_i z_i \\ \sum_i y_i z_i \end{pmatrix}$$

*Extra Credit "Thought Questions" – Answer in 2 or 3 complete sentences:*

A) (5) Exactly why is it the case that a $Z$-test is not appropriate when the sample size is small and population variance is not known exactly? Explain using the case of a test on a single mean.

*Solution: Solution:* The reason is because of the distribution of the *test statistic* we can compute from the data. Under the assumption that the data comes from a normal population, we would estimate $\mu_0$ by $\overline{Y}$ and $\sigma$ by $S$. So we would use

$$t = \frac{\overline{Y} - \mu_0}{\frac{S}{\sqrt{n}}}$$

to carry out this sort of test. If the $S$ was the population SD, $\sigma$, then (under $H_0 : \mu = \mu_0$) this would have a standard normal distribution and the $Z$-test would be appropriate. However, with the sample SD ($S$) estimating $\sigma$, this statistic actually has a $t$-distribution with $n-1$ df, so rejection regions set up using the standard normal would not give enough area in upper and lower tails to account for the variability of $t$.

B) (5) For the basic small-sample test for equality of means, we need to assume the two samples came from normal populations with equal variances. How would you determine whether applying that test was appropriate?

*Solution:* Whether there is a drastic deviation from normality can be detected with normal qq-plots or the Shapiro-Wilk test. Detecting nonequality of variances can be done with an $F$-test. The test statistic $S_1^2/S_2^2$ has an $F$-distribution with $n_1 - 1$ degrees of freedom in the numerator and $n_2 - 1$ degrees of freedom in the denominator under the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$.