Mathematics 375 – Probability and Statistics I
Solutions for Midterm Exam 1 – October 8, 2009

I. To test the effectiveness of a seal for air leaks in automobile tires, after the seal was installed in a tire, a needle was inserted into the tire and air pressure was increased until leakage was observed. The pressures (in lb per square inch) where leakage first occurred on 10 trials were:

A) (10) Construct a relative frequency histogram for this data on the interval $[86.5, 96.5]$, subdividing into 5 equal "bins"

$$(86.5, 88.5], (88.5, 90.5], (90.5, 92.5], (92.5, 94.5], (94.5, 96.5].$$

Solution: The histogram should have a box of height 2/10 on the first interval, a box of height 2/10 on the second interval, a box of height 3/10 on the third interval, a box of height 1/10 on the fourth interval, and a box of height 2/10 on the last interval.

C) (15) How many of the data points are within 1 standard deviation of the sample mean?

Solution: The sample mean is

$$\overline{x} = \frac{93 + 87 + 91 + 90 + 92 + 88 + 95 + 91 + 90 + 96}{10} = 91.3.$$

The sample variance is

$$s^2 = \frac{1}{9}\sum_{i=1}^{10}(x_i - \overline{x})^2 \doteq 8.01.$$

So the sample standard deviation is $s = \sqrt{8.01} \doteq 2.83$. The interval $[\overline{x} - s, \overline{x} + s] = [88.47, 94.13]$. Six of the 10 data values are in this interval. (Note: This is relatively close to the empirical rule of 68%.)

II. Sailors on a ship can send signal by arranging (exactly) 8 colored flags along a rope line.

A) (5) How many different signals can they send if they have flags of 8 different colors?

Solution: This is the number of permutations of 8 things, 8 at a time: 8! (factorial, not exclamation point).

B) (10) How many different signals can they send if they have 4 red, 2 green, and 2 blue flags? (There are no differences between the flags of the same color.)

Solution: The red flags can occur in any 4 of the 8 positions along the rope line. Once the reds are placed, the greens go in 2 of the remaining 4, and then the blues go in the remaining two slots. By the "$m \cdot n$ rule," this gives

$$\binom{8}{4} \cdot \binom{4}{2} \cdot \binom{2}{2} = 420$$

possible arrangements. Note: this is the same as the multinomial coefficient

$$\binom{8}{4\ \ 2\ \ 2}.$$

C) (10) In the situation of part B, if a random arrangement of flags is constructed, what is the probability that at least 5 other flags appear between the two blue flags?

Solution: The answer from part B will be the denominator of the ratio giving the probability we want to compute. The numerator is the number of arrangements of the 8 flags where the two blues are separated by at least 5 other flags. Note that the possible locations for the blue flags are as shown:

$$BXXXXXBX, BXXXXXXB, \text{ or } XBXXXXXB.$$

In each case, the red and green flags can be placed in $\binom{6}{4}\binom{2}{2}$ ways (reasoning as in part B). Therefore the probability we want is

$$\frac{3\binom{6}{4}\binom{2}{2}}{\binom{8}{4}\cdot\binom{4}{2}\cdot\binom{2}{2}} = \frac{3\binom{6}{4}}{\binom{8}{4}\binom{4}{2}}.$$

III. Let A_1, A_2, A_3 be events in a sample space S. Assume that $S = A_1 \cup A_2 \cup A_3$, where $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $P(A_1) = .35$, $P(A_2) = .4$, $P(A_3) = .25$. Finally, let B be another event with $P(B|A_1) = .1$, $P(B|A_2) = .2$ and $P(B|A_3) = .05$.

A) (10) State the Law of Total Probability and use it to compute $P(B)$.

The LTP says that if we have a partition $S = A_1 \cup \cdots \cup A_k$ into events with $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then for all events B,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k).$$

Applying this here:

$$P(B) = (.1)(.35) + (.2)(.4) + (.05)(.25) = .1275.$$

B) (5) Are B and A_2 independent events? Why or why not?

Solution: No. The events B and A_2 are not independent, since $P(B \cap A_2) = P(B|A_2)P(A_2) = .08$ is not the same as $P(B)P(A_2) = (.1275)(.4) = .051$.

C) (10) What is $P(A_1 \cup A_2|B)$? State the name(s) of any general rule(s) you are using.

Solution: Using the definition of conditional probabilities, $P(A_1 \cup A_2|B) = \frac{P((A_1 \cup A_2)\cap B)}{P(B)}$. Using standard properties of set operations, $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$. Since $A_1 \cap A_2 = \emptyset$, it follows that $(A_1 \cap B) \cap (A_2 \cap B) = \emptyset$ also. Therefore,

$P((A_1 \cup A_2) \cap B) = P(A_1 \cap B) + P(A_2 \cap B)$ by the additive rule. It follows by Bayes' Rule that the probability we want is

$$\frac{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}{P(B)} = 1 - \frac{P(B|A_3)}{P(B)} = 1 - (.098) \doteq .902.$$

IV.

A) (10) Let Y have a binomial distribution with parameters $n =$ number of trials and $p =$ success probability on each trial. Show that $E(Y) = np$.

Solution: See the class notes or the text.

B) (15) An army regiment has 20 different squads, 16 with 25 soldiers each, 3 with 100 soldiers each, and one with 300 soldiers, for a total of 1000 soldiers. Select a soldier at random out of the 1000 and let the random variable X equal the size of the squad to which that soldier belongs. Find the probability mass function, the expected value, and the variance of X.

Solution: The possible values with nonzero probabilities are $x = 25, 100, 300$. The probability that X takes one of these values is the probability that the randomly selected soldier belongs to a squad of that size:

$$\begin{aligned} P(X = 25) &= \frac{400}{1000} = .4 \\ P(X = 100) &= \frac{300}{1000} = .3 \\ P(X = 300) &= \frac{300}{1000} = .3 \\ P(X = x) &= 0 \text{ for all other } x. \end{aligned}$$

The expected value is $E(X) = (25)(.4) + (100)(.3) + (300)(.3) = 130$. The variance is

$V(X) = E((X - 130)^2) = (25 - 130)^2(.4) + (100 - 130)^2(.3) + (300 - 130)^2(.3) = 13350.$

(The standard deviation would be more meaningful here: $\sigma = \sqrt{13350} \doteq 115.5$.

Extra Credit (10)

Let X be a discrete random variable which takes as values all natural numbers $k \geq 1$. Assuming that $E(X)$ exists, show that

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k).$$

(Also, why do we need to add the hypothesis that $E(X)$ exists in a case like this?)

Solution: We start from the definition of the expected value and then regroup the terms in the sum (this is justified since the infinite series must be absolutely convergent – it has all non-negative terms!) :

$$\begin{aligned}
E(X) &= \sum_{k=1}^{\infty} kP(X = k) \\
&= P(X = 1) + 2P(x = 2) + 3P(X = 3) + \cdots \\
&= P(X = 1) + P(X = 2) + P(X = 3) + \cdots \\
&\qquad\qquad\qquad\;\; + P(X = 2) + P(X = 3) + \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\;\; + P(X = 3) + \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \cdots \\
&= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots \\
&= \sum_{k=1}^{\infty} P(X \geq k).
\end{aligned}$$

The hypothesis on $E(X)$ is necessary for this kind of random variable since for some other Y, $\sum_{y=1}^{\infty} P(Y = y)$ would exist and sum to 1, but the series $\sum_{y=1}^{\infty} yP(Y = y)$ might *diverge*. An example would be a random variable with

$$P(Y = y) = \frac{6}{\pi^2}\frac{1}{y^2}.$$

The series $\sum_{y=1}^{\infty} \frac{1}{y^2}$ converges (p-series with $p = 2$). The sum is actually $\frac{\pi^2}{6}$, and that accounts for the normalization factor in the formula for $P(Y = y)$ above. But $\sum_{y=1} y\frac{1}{y^2} = \sum_{y=1}^{\infty} \frac{1}{y}$ diverges (harmonic series).