Mathematics 375 – Probability and Statistics 1
Solutions for Problem Set 3
September 24, 2009

2.71. (a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{3}$.
(b) $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1}{5}$
(c) We have $P(A \cup B) = P(A) + P(B) - P(A\cap B) = .7$. Therefore, $P(A|A\cup B) = \frac{P(A)}{P(A\cup B)} = \frac{5}{7}$.
(d) $P(A|A\cap B) = 1$, since $(A\cap B)\cap A = A\cap B$.
(e) $P(A\cap B|A\cup B) = \frac{1}{7}$.

2.72. (a) $P(A\cap M) = .24$ while $P(A) = .6$ and $P(M) = .4$. Since $P(A\cap M) = P(A)P(M)$, the events A, M *are independent* (see Definition 2.10).
(b) $P(\overline{A}\cap F) = .24$ while $P(\overline{A}) = .4$ and $P(F) = .6$. Since $P(\overline{A}\cap F) = P(\overline{A})P(F)$, these events are also independent.

2.74. (a) $P(A\cap D) = .2$, while $P(A) = .61$ and $P(D) = .3$ since $.2 \neq (.61)(.3)$, A and D are *not independent*.
(b) $P(B\cap D) = .09 = P(B)P(D) = (.3)(.3)$. So B and D *are independent*.
(c) $P(C\cap D) = .01$, but $P(C)P(D) = (.09)(.3) = .027$. So C and D are *not independent*.

2.75. (a) Let A be the event that the first two cards dealt are spades, so $P(A) = \frac{13}{52}\cdot\frac{12}{51} = \frac{1}{17}$. Next, let B be the event that cards $3, 4, 5$ are also spades. We want

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}\cdot\frac{9}{48}}{\frac{13}{52}\cdot\frac{12}{51}} = \frac{33}{3920} \doteq .008.$$

(b) Similarly, let C be the event that the first three cards are spades, and D be the event that cards $4, 5$ are spades.

$$P(D|C) = \frac{P(D\cap C)}{P(C)} = \frac{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}\cdot\frac{9}{48}}{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}} = \frac{15}{392} \doteq .04.$$

(c) Similarly, let F be the event that the first four cards are spades, and F be the envent that card 5 is a spade.

$$P(F|E) = \frac{P(F\cap E)}{P(E)} = \frac{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}\cdot\frac{9}{48}}{\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}} = \frac{9}{48} \doteq .19.$$

2.81. We are given that $P(A), P(B) > 0$ and $P(A) < P(A|B)$. By the definition of conditional probability, this last statement says $P(A) < \frac{P(A\cap B)}{P(B)}$, so since $P(B) > 0$, $P(A)P(B) < P(A\cap B)$. Now divide this statement through by $P(A)$: $P(B) < \frac{P(A\cap B)}{P(A)}$. But this says $P(B) < P(B|A)$, which is what we wanted to show.

2.84. "At least one A_i" means $A_1 \cup A_2 \cup A_3$. The general formula for the probability of a 3-fold union is given in the text on page 58. In the terms on the right, we know $P(A_2 \cap A_3) = 0$, and it follows that $P(A_1 \cap A_2 \cap A_3) = 0$, since $A_1 \cap A_2 \cap A_3 \subset A_2 \cap A_3$. Moreover, we are given that $P(A_1 \cap A_2) = P(A_1 \cap A_3)$. The required formula follows.

2.85. Since $S = B \cup \overline{B}$ and $B \cap \overline{B} = \emptyset$, we have $A = (A \cap B) \cup (A \cap \overline{B})$ and $P(A) = P(A \cap B) + P(A \cap \overline{B})$. Since A and B are independent, $P(A \cap B) = P(A)P(B)$. Substituting for $P(A)$, we obtain:

$$P(A \cap B) = (P(A \cap B) + P(A \cap \overline{B}))P(B),$$

so after algebraic rearrangement

$$P(A \cap B)(1 - P(B)) = P(A \cap \overline{B})P(B).$$

Since $P(A \cap B) = P(A)P(B)$ again, this simplifies to

$$P(A)P(\overline{B}) = P(A \cap \overline{B}),$$

which shows that A and $\overline{B}$ are independent. The answer is yes for the second question. From $P(\overline{A} \cap \overline{B}) + P(A \cap \overline{B}) = P(\overline{B})$,

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= P(\overline{B}) - P(A \cap \overline{B}) \\ &= P(\overline{B}) - P(A)P(\overline{B}) \quad \text{by the first part} \\ &= P(\overline{B})(1 - P(A)) \\ &= P(\overline{B})P(\overline{A}) \end{aligned}$$

This shows $\overline{A}$ and $\overline{B}$ are independent.

2.88. (a) It is possible for $P(A \cap B) = .1$ if $P(A \cup B) = .8$.
(b) The smallest possible value for $P(A \cap B)$ is 0 (i.e. A and B can be disjoint).
(c) It is *not possible* for $P(A \cap B) = .7$, since $A \cap B \subset A$ and $P(A) = .6$.
(d) The largest possible value for $P(A \cap B)$ is .3 (i.e. B can be contained in A, so $A \cap B = B$ is possible.)

2.95. (a) From the additive law (Theorem 2.6), $.4 = .2 + .3 - P(A \cap B)$, so $P(A \cap B) = .1$.
(b) By DeMorgan's Laws, $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - .1 = .9$.
(c) Similarly, $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - .4 = .6$.
(d) $P(\overline{A}|B) = \frac{P(\overline{A} \cap B)}{P(B)}$. The numerator can be computed since we know $P(A \cap B) = .1$ and $P(A \cap B) + P(\overline{A} \cap B) = P(B) = .3$. So the probability we want is $\frac{.2}{.3} = \frac{2}{3}$.

2.97. (a) The probability of current flowing is

$$\begin{aligned} P(1 \cup 2 \cup 3) &= P(1) + P(2) + P(3) - P(1 \cap 2) - P(1 \cap 3) - P(2 \cap 3) + P(1 \cap 2 \cap 3) \\ &= .9 + .9 + .9 - .81 - .81 - .81 + .729 \\ &= .999 \end{aligned}$$

(Alternate method: The only way current can fail to flow is if all three relays fail, which happens with probability $(.1)^3 = .001$. So the complementary probability is .999.)

(b) If relay 1 functioned properly, then current did flow, so $P(\text{flow} \cap 1) = P(1) = .9$. Hence $P(1|\text{flow}) = \frac{.9}{.999} = .\overline{900}$.

Additional Problem

(2) For large numbers of runs (say ≥ 10000), the fraction $\frac{\text{wins}}{\text{total games}}$ should be roughly .493.

(3) a) The probability of winning immediately is the probability of rolling a total of 7 or 11, which is $\frac{6}{36} + \frac{2}{36} = \frac{2}{9}$. The probability of losing immediately is the probability of rolling a 2, 3, or 12, which is $\frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$.

(b) Let A be the event that you win, $F4$ be the event that you roll a 4 on the first roll. The conditional probability $P(A|F4)$ can be analyzed like this: $A \cap F4$ is the union of "subevents" A_k:

$$\begin{cases} A_2 = 4 & \text{is rolled on 2nd roll,} \\ A_3 = 4 & \text{is rolled on 3rd roll and second roll was not a 7,} \\ A_4 = 4 & \text{is rolled on 4th roll and second and third rolls are not 7's,} \\ & \vdots \end{cases}$$

These are mutually exclusive (no overlaps), but any number of rolls can happen before the next 4 comes up. Hence

$$P(A \cap F4) = \sum_{k=2}^{\infty} P(A_k) = \sum_{k=2}^{\infty} \left(\frac{3}{36}\right)\left(\frac{27}{36}\right)^{k-2}$$

(the $\frac{3}{36}$ is the probability of rolling a 4; the $\frac{2}{36}$ is the probability of rolling anything other than a 4 or a 7).

Then reasoning similarly, let $F5, F6, F8, F9, F10$ be the events corresponding to the different first rolls in this case.

$$\begin{aligned} P(A \cap F10) &= P(A \cap F4) \\ P(A \cap F5) = P(A \cap F9) &= \sum_{k=2}^{\infty} P(A_k) = \sum_{k=2}^{\infty} \left(\frac{4}{36}\right)\left(\frac{26}{36}\right)^{k-2} \\ P(A \cap F6) = P(A \cap F8) &= \sum_{k=2}^{\infty} P(A_k) = \sum_{k=2}^{\infty} \left(\frac{5}{36}\right)\left(\frac{25}{36}\right)^{k-2} \end{aligned}$$

Then by the Law of Total Probability

$$\begin{aligned} P(A) &= P(\text{win immediately}) + P(A|F4)P(F4) + P(A|F5)P(F5) + P(A|F6)P(F6) \\ &\quad + P(A|F8)P(F8) + P(A|F9)P(F9) + P(A|F10)P(F10) \\ &= \frac{2}{9} + 2\sum_{k=2}^{\infty}\left(\frac{3}{36}\right)^2\left(\frac{27}{36}\right)^{k-2} + 2\sum_{k=2}^{\infty}\left(\frac{4}{36}\right)^2\left(\frac{26}{36}\right)^{k-2} + 2\sum_{k=2}^{\infty}\left(\frac{5}{36}\right)^2\left(\frac{25}{36}\right)^{k-2} \\ &= \frac{244}{495} \doteq .49283 \end{aligned}$$

(The sums are *geometric series* and can be summed by the usual formula:

$$\sum_{\ell=0}^{\infty} cr^{\ell} = \frac{c}{1-r}.)$$