Mathematics 375 – Probability and Statistics I
Midterm Exam 1 – Solutions

I. To test the effectiveness of a seal designed to keep an electrical plug airtight, a needle was inserted into the plug and air pressure was increased until leakage was observed. The pressures (in lb per square inch) where leakage first occurred on 10 trials were:

$$3.1, 3.5, 3.3, 4.5, 4.2, 2.8, 3.9, 3.5, 3.3, 4.0$$

A) (10) Construct a relative frequency histogram for this data on the interval $[3, 5]$, subdividing into 5 equal "bins".

Solution: With 5 equal "bins" $\Delta x = (5 - 3)/5 = .4$. If a measurement falls on a bin boundary, we can consistently place it in the lower bin. Then there are 3 measurements in $[3, 3.4]$, 2 in $(3.4, 3.8]$, 3 in $(3.8, 4.2]$, 1 in $(4.2, 4.6]$ and none greater than 4.6. The relative frequency histogram has one box of height 3/10 over the first interval, one of height 2/10 over the second, one of height 3/10 over the third, one of height 1/10 over the fourth, and one of height 0 over the last. There is one datapoint out of this range.

B) (5) What is the sample mean?

Solution:

$$\overline{x} = \frac{1}{10}(3.1 + 3.5 + 3.3 + 4.5 + 4.2 + 2.8 + 3.9 + 3.5 + 3.3 + 4.0) = 3.61$$

C) (10) How many of the data points are within 2 standard deviations of the mean?

Solution: First we need to compute

$$s = \sqrt{\frac{1}{9}\sum_{i=1}^{10}(x_i - \overline{x})^2} = \sqrt{.279} = .528$$

"Within 2 standard deviations of the mean" means in the interval $[\overline{x} - 2s, \overline{x} + 2s] = [2.554, 4.666]$. All 10 of the measurements are in this interval.

II. Students on a boat send signals back to shore by arranging (exactly) 9 colored flags on a vertical flagpole.

A) (5) How many different signals can they send if they have flags of 9 different colors?

Solution: This is the number of permutations of 9 things taken 9 at a time: $P_9^9 = 9!$.

B) (10) How many different signals can they send if they have 4 green, 3 red, and 2 blue flags? (There are no differences between the flags of the same color.)

Solution: The 4 green flags can go in any 4 of the 9 locations, so there are $\binom{9}{4}$ ways to place them. Then once the greens have been placed, there are 5 slots left for the 3 red flags, so there are $\binom{5}{3}$ ways to place them. There are $\binom{2}{2}$ ways to place the blues after the greens and reds have been placed. This gives

$$\binom{9}{4}\binom{5}{3}\binom{2}{2} = \frac{9!}{4!3!2!} = 1260$$

different ways.

C) (10) In the situation of part B, if a random arrangement of flags is constructed, what is the probability that all the green flags appear *consecutively*?

Solution: The string of consecutive green flags can start in locations 1,2,3,4,5, or 6 along the pole. For each of these, there are then $\binom{5}{3}\binom{2}{2}$ ways left to place the reds and the blues. (Those *do not* need to be consecutive, of course!). If a random arrangement is chosen, any arrangement is equally likely, and the probability that the green flags are consecutive is

$$\frac{6 \cdot \binom{5}{3}\binom{2}{2}}{\binom{9}{4}\binom{5}{3}\binom{2}{2}} = \frac{6}{\binom{9}{4}} \doteq .048.$$

III. Let A_1, A_2, A_3 be events in a sample space S. Assume that $S = A_1 \cup A_2 \cup A_3$, where $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $P(A_1) = .3$, $P(A_2) = .2$, $P(A_3) = .5$. Finally, let B be another event with $P(B|A_1) = .1$, $P(B|A_2) = .2$ and $P(B|A_3) = .05$.

A) (10) What is $P(B)$?

Solution: We use the Law of Total Probability:

$$P(B) = \sum_{i=1}^{3} P(B|A_i)P(A_i) = (.1)(.3) + (.2)(.2) + (.05)(.5) = .095.$$

B) (5) Are B and A_2 independent events? Why or why not?

Solution: B and A_2 are *not* independent events because $P(B) = .095$, but $P(B|A_2) = .2$, so $P(B|A_2) \neq P(B)$.

C) (10) Find $P(A_1 \cup A_2|B)$.

Solution: Since $A_1 \cap A_2 = \emptyset$,

$$P((A_1 \cup A_2) \cap B) = P((A_1 \cap B) \cup (A_2 \cap B)) = P(A_1 \cap B) + P(A_2 \cap B).$$

Hence

$$P(A_1 \cup A_2|B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$

By Bayes' Rule, then

$$P(A_1|B) + P(A_2|B) = \frac{P(B|A_1)P(A_1)}{.095} + \frac{P(B|A_2)P(A_2)}{.095} = \frac{(.1)(.3) + (.2)(.2)}{.095} = .737.$$

IV.

A) (15) A jar contains 7 red and 11 white balls. Draw *two balls* at random, *without replacing the first ball after it is drawn.* Let X be the number of balls drawn that are red. Thinking of X as a discrete random variable, find its probability mass function, and compute its expected value and variance.

Solution: This was the one slightly tricky question on this exam. X does not have a binomial distribution because when you remove a ball from the jar the probabilities of drawing each color on the second draw *change.* We have

x	$P(X = x)$
0	$\frac{11}{18}\frac{10}{17} = \frac{110}{306} = .359$
1	$2\frac{11}{18}\frac{7}{17} = \frac{154}{306} = .503$
2	$\frac{7}{18}\frac{6}{17} = \frac{42}{306} = .137$

(Note: these don't sum to exactly 1 because of rounding.)

Then

$$E(X) = 0(.359) + 1(.503) + 2(.137) = .778$$

and

$$E(X^2) = 0(.359) + 1(.503) + 4(.137) = 1.052$$

so $V(X) = E(X^2) - (E(X))^2 = 1.052 - (.778)^2 = .447.$

B) (10) Assume you have a 40% of connecting each time you dial a very busy customer service telephone line. If 25 calls are made at random and independently, what is the probability that between 15 and 20 of them (inclusive) will get through?

Solution: This one *is* binomial, with $n = 25$ and $p = .40$. From the binomial table,

$$P(15 \leq Y \leq 20) = P(Y \leq 20) - P(Y \leq 14) = 1.000 - .966 = .034$$

Extra Credit (10) Let Y have a geometric distribution with success probability p. Show that the expected value of $g(Y) = e^Y$ is $E(e^Y) = \frac{pe}{1-qe}$.

Solution: By the geometric series sum formula,

$$E(e^Y) = \sum_{y=1}^{\infty} e^y q^{y-1} p = \frac{p}{q}\sum_{y=1}^{\infty}(qe)^y = \frac{pqe}{q(1-qe)} = \frac{pe}{1-qe}.$$