

Mathematics 242 – Principles of Analysis
Solutions for Problem Set 9 – **Due:** Friday, April 25

‘A’ Section

1. Let $f(x) = x^2 + 4x - 1$ on $[2, 3]$.

- (a) Show that f is integrable on $[2, 3]$ directly using the definition (that is, do not use any general theorems giving criteria for integrability). Hints: Use regular partitions of $[2, 3]$, and the summation rules

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: Consider the regular partition \mathcal{P}_n of the interval $[2, 3]$, so $x_i = 2 + \frac{i}{n}$ for $i = 0, 1, \dots, n$. We have $f'(x) = 2x + 4 > 0$ for all $x \in [2, 3]$. Hence f is increasing on the interval and thus m_i comes from evaluating at the left endpoints and M_i comes from evaluating at the right endpoints:

$$\begin{aligned} L(f, \mathcal{P}_n) &= \sum_{i=0}^{n-1} \left(\left(2 + \frac{i}{n} \right)^2 + 4 \left(2 + \frac{i}{n} \right) - 1 \right) \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{i=0}^{n-1} i^2 + \frac{8}{n^2} \sum_{i=0}^{n-1} i + \frac{11}{n} \sum_{i=0}^{n-1} 1 \\ &= \frac{(n-1)n(2n-1)}{6n^3} + \frac{8(n-1)n}{2n^2} + 11. \end{aligned}$$

Similarly, evaluating at the right endpoints,

$$\begin{aligned} U(f, \mathcal{P}_n) &= \sum_{i=1}^n \left(\left(2 + \frac{i}{n} \right)^2 + 4 \left(2 + \frac{i}{n} \right) - 1 \right) \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{i=1}^n i^2 + \frac{8}{n^2} \sum_{i=1}^n i + \frac{11}{n} \sum_{i=1}^n 1 \\ &= \frac{n(n+1)(2n+1)}{6n^3} + \frac{8n(n+1)}{2n^2} + 11. \end{aligned}$$

Hence

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{9}{n}.$$

We can make this $< \varepsilon$ for any $\varepsilon > 0$ by letting $n > \frac{9}{\varepsilon}$. Hence f is integrable by our definition.

- (b) Determine the value of $\int_2^3 x^2 + 4x - 1 \, dx$.

Solution: The value of the integral is

$$\lim_{n \rightarrow \infty} U(f, \mathcal{P}_n) = \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3} + \frac{8n(n+1)}{2n^2} + 11 = \frac{1}{3} + \frac{8}{2} + 11 = \frac{46}{3}.$$

(This can be checked by the FTC:

$$\begin{aligned} \int_2^3 x^2 + 4x - 1 \, dx &= \left. \frac{x^3}{3} + \frac{4x^2}{2} - x \right|_1^2 \\ &= \left(\frac{27}{3} - \frac{8}{3} \right) + (18 - 8) + (3 - 2) \\ &= \frac{46}{3}. \end{aligned}$$

2. Let $F(x) = \int_0^x f(t) \, dt$ where

$$f(t) = \begin{cases} 1 - t & \text{if } 0 \leq t \leq 1 \\ 1 + t & \text{if } 1 < t \leq 2 \end{cases}$$

(a) Find an explicit formula for $F(x)$ valid for all $0 \leq x \leq 2$. (Use the FTC in an appropriate way for this.)

Solution: Using the FTC, we have

$$F(x) = t - \frac{t^2}{2} \Big|_0^x = x - \frac{x^2}{2}$$

if $0 \leq x \leq 1$. Note that the value at $x = 1$ is $F(1) = \frac{1}{2}$. Then for x between 1 and 2, we want an antiderivative that takes the value $\frac{1}{2}$ at $x = 1$, so

$$F(x) = x + \frac{x^2}{2} - 1$$

for $1 \leq x \leq 2$. Therefore, putting these together:

$$F(x) = \begin{cases} x - \frac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ \frac{x^2}{2} + x - 1 & \text{if } 1 < x \leq 2 \end{cases}$$

(The constant term in the formula on the second half of the interval ensures that F is continuous on $[0, 2]$.)

(b) Is F differentiable at $x = 1$? Why or why not? Does this contradict the first part of the FTC?

Solution: No, F is not differentiable at $x = 1$, since

$$\lim_{x \rightarrow 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\frac{x^2}{2} - x + \frac{1}{2}}{x - 1} = \lim_{x \rightarrow 1^-} \frac{1}{2}x - \frac{1}{2} = 0,$$

but

$$\lim_{x \rightarrow 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{\frac{x^2}{2} + x - \frac{3}{2}}{x - 1} = \lim_{x \rightarrow 1^+} \frac{1}{2}(x + 3) = 2.$$

This does not contradict the first part of the FTC since f is not continuous on $[0, 2]$. It has a jump discontinuity at $x = 1$ since $\lim_{t \rightarrow 1^-} f(t) = 0$, but $\lim_{t \rightarrow 1^+} f(t) = 2$.

3. Find the following derivatives using the FTC (and other derivative rules as needed):

- (a) $\frac{d}{dx} \int_1^x \frac{e^t}{t} dt$ (Note that the integrand is not defined at $t = 0$, so 0 should not be included in the interval of integration, or else the FTC does not apply.)

Solution: The first part of the FTC implies the derivative is $\frac{e^x}{x}$.

- (b) $\frac{d}{dx} \int_{-x^2}^{x^4} \cos(e^u) du$

Solution: By the first part of the FTC and the chain rule, the derivative is

$$4x^3 \cos(e^{x^4}) + 2x \cos(e^{-x^2}).$$

'B' Section

1. Let f be integrable on the interval $[a, b]$, and assume $\int_a^b f(x) dx < 0$. Show that there exist $k < 0$ and an interval $[c, d] \subseteq [a, b]$ such that $f(x) < k < 0$ for all $x \in [c, d]$.

Solution: If the function is integrable, then the definition of integrability implies that for every $\varepsilon > 0$, there exists a partition \mathcal{P} such that

$$(1) \quad U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon.$$

Moreover, it is always true that

$$(2) \quad L(f, \mathcal{P}) \leq \int_a^b f(x) dx \leq U(f, \mathcal{P}).$$

Since the integral is strictly negative, this means that if we take $0 < \varepsilon < \left| \int_a^b f(x) dx \right|$, it follows from (1) and (2) that

$$U(f, \mathcal{P}) < L(f, \mathcal{P}) + \varepsilon < \int_a^b f(x) dx + \left| \int_a^b f(x) dx \right| = 0.$$

In other words, the lower sum $U(f, \mathcal{P}) < 0$ for some partition. This can only happen if $M_i = \text{glb}\{f(x) \mid x \in [x_{i-1}, x_i]\} < 0$ over some interval $[x_{i-1}, x_i] \subset [a, b]$ from the partition \mathcal{P} . Hence $f(x) \leq M_i < M_i/2 < 0$ for all $x \in [c, d] = [x_{i-1}, x_i]$. We can take $k = m/2$ to get the strict inequality desired.

2. Let f be continuous on $[a, b]$. Show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x) dx = f(c)(b - a).$$

(Hint: Look at Theorem 5.2.5 (b).)

Solution: Since f is continuous on $[a, b]$, by the Extreme Value Theorem, it attains a maximum M and a minimum m on the interval. By the theorem indicated,

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a),$$

so

$$m \leq \frac{1}{b - a} \int_a^b f(x) dx \leq M.$$

But then by the Intermediate Value Theorem (in the second version we discussed), it follows that

$$\frac{1}{b - a} \int_a^b f(x) dx = f(c)$$

for some $c \in [a, b]$. Hence after multiplying through again by $b - a$, we get:

$$\int_a^b f(x) dx = f(c)(b - a)$$

for some $c \in [a, b]$. (Comment: this result is sometimes called the Mean Value Theorem for Integrals. It is related to the method of computing the *average value* of a function f on $[a, b]$ that you may have seen as an application of integration in Calculus 2.)

3. Logarithm and Exponential. In this problem, we will construct the natural logarithm and the exponential function e^x “from scratch,” without relying on intuition about exponentials (as you probably did in calculus). We start by considering the function

$$(1) \quad L(x) = \int_1^x \frac{1}{t} dt,$$

for $x > 0$. Note that $\frac{1}{t}$ is continuous on $(0, +\infty)$, hence the FTC applies to show that L is a differentiable function. From calculus you probably recognize that $L(x) = \ln(x)$. We want to show directly that this makes sense and use this function to construct the inverse function $E(x) = L^{-1}(x)$ which is called $E(x) = e^x$. Why would we proceed “backwards”

like this? The issue is that, while $a^{m/n} = (a^{1/n})^m$ makes immediate sense for any positive $a \in \mathbf{R}$ and any rational exponent, what does a^x actually mean if $x \notin \mathbf{Q}$? Instead of trying to define that directly, we will take an end run around the question.

- (a) We claim that the function L “has the right property to be a logarithm” – namely that $L(x \cdot x') = L(x) + L(x')$ for all $x, x' > 0$. Prove this by showing that for any constant $x' > 0$, the function $L(xx')$ also satisfies $\frac{d}{dx}L(xx') = \frac{1}{x}$ for all $x > 0$. Deduce that $L(xx') = L(x) + c$ for some constant c , then determine c by substituting an appropriate value for x .

Solution: By the first part of the FTC and the chain rule, we have $\frac{d}{dx}L(xx') = \frac{1}{xx'} \cdot x' = \frac{1}{x}$. Since $L(x)$ also satisfies $L'(x) = \frac{1}{x}$, this implies by one of our corollaries of the Mean Value Theorem that $L(xx') - L(x) = c$ is a constant. By definition $L(1) = 0$, so if we substitute $x = 1$, we get $c = L(x')$. Therefore $L(xx') = L(x) + L(x')$.

- (b) Show that $L(x)$ is strictly increasing for $x > 0$, hence is a 1-1 function on the domain $(0, +\infty)$. Hence L has an inverse function $E : \mathbf{R} \rightarrow \{x \in \mathbf{R} \mid x > 0\}$.

Solution: Most of this follows since $L'(x) = \frac{1}{x} > 0$ on $(0, +\infty)$, by the first part of the FTC. Hence L is strictly increasing by one of the corollaries of the MVT, hence 1-1.

Comment: To give every detail here, we should also show that the range of L is \mathbf{R} to show that the domain of the inverse function is actually equal to \mathbf{R} . To see this, we can “borrow” from the fact that the harmonic series is divergent, like this: We want to show first that $\{L(x) \mid x \in (0, \infty)\}$ is not bounded above. Let $x = n$ be an integer $n > 1$. By considering the lower sum of $\int_1^n \frac{dt}{t}$ for the partition $\mathcal{P} = \{1, 2, \dots, n\}$ for the interval $[1, n]$, we can see that

$$L(n) = \int_1^n \frac{dt}{t} > \frac{1}{2} + \dots + \frac{1}{n} = L\left(\frac{1}{t}, \mathcal{P}\right).$$

As $n \rightarrow \infty$, though, the right side increases without any bound, since the harmonic series diverges. Hence the numbers $L(n)$ have no upper bound either. On the other hand by part (a) we have

$$0 = L(1) = L\left(n \cdot \frac{1}{n}\right) = L(n) + L\left(\frac{1}{n}\right).$$

Hence the values $L\left(\frac{1}{n}\right)$ are not bounded below either (they go to $-\infty$ as $n \rightarrow \infty$). Since L is continuous, its range must equal $(-\infty, \infty) = \mathbf{R}$ (by an application of the Intermediate Value Theorem – do you see why?)

- (c) Show that the inverse function E satisfies the equation $E(x+x') = E(x) \cdot E(x')$, hence E looks like an exponential function.

Solution: By the definition of an inverse function we have

$$y = E(x) \Leftrightarrow x = L(y).$$

Let y, y' be any two positive real numbers and let $x = L(y)$, $x' = L(y')$, so $y = E(x)$ and $y' = E(x')$. By part (a) we know that

$$L(yy') = L(y) + L(y').$$

Hence

$$E(x)E(x') = yy' = E(L(y) + L(y')) = E(x + x').$$

- (d) We define $x = e$ as the unique solution of the equation $L(x) = 1$. Show that $x = e^{m/n}$ satisfies $L(x) = m/n$ for all rational numbers $x = m/n$.

Solution: The given information says $L(e) = 1$. First we claim $L(e^m) = m$ for all $m \in \mathbf{N}$. This follows by induction since $L(e^1) = 1$ by definition, and if $L(e^k) = k$, then $L(e^{k+1}) = L(e^k \cdot e) = L(e^k) + L(e) = k + 1$ by part (a) and the induction hypothesis.

Next, since $L(e^m) + L(e^{-m}) = L(e^m \cdot e^{-m}) = L(1) = 0$, we have $L(e^{-m}) = -m$, which shows that $L(e^m) = m$ for all integers m . Next, by part (a) again, if n is a positive integer, $1 = L(e) = L((e^{1/n})^n) = nL(e^{1/n})$. So $L(e^{1/n}) = \frac{1}{n}$ whenever n is a positive integer. Finally, combining all of these steps we can write any rational number m/n as a fraction in which $n > 0$ and then

$$L(e^{m/n}) = L((e^{1/n})^m) = mL(e^{1/n}) = \frac{m}{n}L(e) = \frac{m}{n}.$$

- (e) Show that the inverse function $E(x)$ of $L(x)$ is differentiable and satisfies $E'(x) = E(x)$. Hint: Look at Section 4.4. What happens if you differentiate on both sides of the equation $L(E(x)) = x$?

Solution: Theorem 4.4.4 implies that E is differentiable at all real x , since $L'(x) = \frac{1}{x}$ is never equal to zero. Hence if we differentiate the equation $L(E(x)) = x$ on both sides using the chain rule we get $L'(E(x))E'(x) = 1$. But $L'(u) = \frac{1}{u}$, so this says $\frac{1}{E(x)} \cdot E'(x) = 1$ for all x , so $E'(x) = E(x)$. This is the familiar differentiation rule for the exponential function: $\frac{d}{dx}e^x = e^x$.

- (f) Show that

$$E(x) = e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

for all $x \in \mathbf{R}$. Hint: Take the logarithm and use L'Hopital's Rule.

Solution: We have

$$L\left(\left(1 + \frac{x}{n}\right)^n\right) = nL\left(1 + \frac{x}{n}\right) = \frac{L\left(1 + \frac{x}{n}\right)}{\frac{1}{n}}.$$

This is the value of a ratio $g(u)/h(u)$ at $u = \frac{1}{n}$. The top is $g(u) = L(1 + ux)$ and the bottom is $h(u) = u$. Both of these are “nice functions” differentiable at $u = 0$. Note that $\lim_{u \rightarrow 0} g(u) = 0 = \lim_{u \rightarrow 0} h(u)$. So we can apply the version of L’Hopital’s Rule developed on Problem Set 8. We have $g'(u) = \frac{x}{1+ux}$ and $h'(u) = 1$. The ratio $g'(u)/h'(u)$ does have a limit as $u \rightarrow 0$:

$$\lim_{u \rightarrow 0} \frac{g'(u)}{h'(u)} = \lim_{u \rightarrow 0} \frac{x}{1 + ux} = x.$$

Therefore by L’Hopital’s Rule we know

$$\lim_{u \rightarrow 0} \frac{g(u)}{h(u)} = x$$

as well. Since $u = \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$, this implies

$$\lim_{n \rightarrow \infty} \frac{L\left(1 + \frac{x}{n}\right)}{\frac{1}{n}} = x.$$

Since L is differentiable function at all $x > 0$, it also continuous everywhere. Hence

$$L\left(\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n\right) = \lim_{n \rightarrow \infty} L\left(\left(1 + \frac{x}{n}\right)^n\right) = x$$

This implies

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = E(x) = e^x.$$