Mathematics 242 – Principles of Analysis
Solutions for Problem Set 8 – **Due:** Friday, April 19

'A' Section

1. For each of the following functions and intervals: First state whether the hypotheses of the MVT are satisfied for f. Second: If they are satisfied, determine all $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. If they are not satisfied, determine whether the conclusion of the MVT holds even so.

(a) $f(x) = \frac{4x+1}{x-1}$ on $[a,b] = [0,2]$.

Solution: The equation $x - 1 = 0$ has a solution in $(0,2)$. Hence the MVT does not apply, since f is not differentiable at every $x \in (0,2)$. However the conclusion of the MVT could still hold so we need to check. We have $f(0) = -1$, $f(2) = 9$, so

$$\frac{f(2) - f(0)}{2 - 0} = \frac{10}{2} = 5.$$

By the quotient rule, we have

$$f'(x) = \frac{(x-1)(4) - (4x+1)(1)}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

This is always negative where it is defined, so there is no c where $f'(c) = 5$.

(b) $f(x) = x^{1/3}$ on $[a,b] = [0,1]$.

Solution: f is continuous on $[0,1]$ and differentiable on $(0,1)$ with $f'(x) = \frac{1}{3x^{2/3}}$. Hence the MVT does apply. We have

$$\frac{1}{3c^{2/3}} = \frac{\sqrt{1} - \sqrt{0}}{1 - 0} = 1$$

when $c^{2/3} = \frac{1}{3}$, or $c = \frac{1}{3\sqrt{3}}$.

(c) $f(x) = \sin(\pi x)$ on $[0,3]$.

Solution: Using problem 1 on the 'B' section, f is differentiable at all x, hence continuous on $[0,3]$. The hypotheses of the MVT are satisfied. By the derivative formula for $\sin(x)$ and the chain rule we have $f'(x) = \pi\cos(\pi x)$. Since $f(3) = 0$ and $f(0) = 0$, we are looking for $c \in (0,3)$ where

$$\pi\cos(\pi c) = 0, \text{ so } \cos(\pi c) = 0$$

There are 3 such values of c in the interval $(0,3)$, at $c = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

(d) $f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 2 & \text{if } x = 0 \end{cases}$ on $[a, b] = [0, 2]$.

Solution: f is not continuous at 0 so the hypotheses of the MVT are not satisfied. $\frac{f(2)-f(0)}{2-0} = \frac{-3}{4}$. For $x > 0$, $f'(x) = \frac{-1}{x^2}$. So there is a $c \in (0, 2)$ where

$$\frac{-1}{c^2} = \frac{-3}{4},$$

namely $c = 2/\sqrt{3} \doteq 1.1547$. Examples like this one show that even when the hypotheses of the MVT are not satisfied, the conclusion can still be true.

2. Let $f(x) = x^3 - \lambda x^2 - \lambda^2 x$, where $\lambda > 0$ is a constant.

(a) Show that f is a 1-1 function on the interval $I = [-\lambda/3, \lambda]$.

Solution: We have $f'(x) = 3x^2 - 2\lambda x - \lambda^2 = (3x + \lambda)(x - \lambda)$, which is strictly negative on the open interval $(-\lambda/3, \lambda)$. Hence by one of the corollaries of the MVT (Theorem 4.3.15 in the text), f is strictly decreasing on the interval $[-\lambda/3, \lambda]$.

(b) Determine $f(I)$.

Solution: $f(-\lambda/3) = 5\lambda^3/27$ and $f(\lambda) = -\lambda^3$. So the image is $\left[-\lambda^3, \frac{5\lambda^3}{27}\right]$ using the Intermediate Value Theorem.

(c) Let g be the inverse function of f restricted to I. What is $g'(0)$?

Solution: We have $g(f(x)) = x$ for all $x \in I$. By the Chain Rule, $g'(f(x))f'(x) = 1$. Since f is 1-1 on I, there is just one x where $f(x) = 0$ for $x \in I$, namely $x = 0$. Then $f'(0) = -\lambda^2$. Hence $g'(0) = \frac{1}{f'(0)} = \frac{-1}{\lambda^2}$.

3. Let $f(x) = x^2 + x + 3$ on $[0, 3]$.

(a) Compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$ for the partition $\mathcal{P} = \{0, 1, 2, 3\}$.

Solution: We have $f'(x) = 2x + 1 > 0$ on $[0, 3]$, f is strictly increasing on this interval. Therefore $m_1 = f(0) = 3$, $M_1 = f(1) = 5$, $m_2 = f(1) = 5$, $M_2 = f(2) = 9$, $m_3 = f(2) = 9$ and $M_3 = f(3) = 15$. Hence

$$L(f, \mathcal{P}) = (3)(1) + (5)(1) + (9)(1) = 17$$
$$U(f, \mathcal{P}) = (5)(1) + (9)(1) + (15)(1) = 29$$

(b) Compute $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$ for the partition $\mathcal{P}' = \{0, 1/2, 1, 2, 5/2, 3\}$ and verify that the statement of Lemma 5.1.9 holds here.

Solution: For the partition $\mathcal{P}'$, which is a refinement of $\mathcal{P}$, we have $m_1 = 3$, $M_1 = f(1/2) = 15/4$, $m_2 = 15/4$, $M_2 = f(1) = 5$, $m_3 = 5$, $M_3 = f(2) = 9$, $m_4 = 9$, $M_4 = f(5/2) = 47/4$, $m_5 = 47/4$, $M_5 = 15$. Hence

$$L(f, \mathcal{P}') = (3)(1/2) + (15/4)(1/2) + (5)(1) + (9)(1/2) + (47/4)(1/2) = 75/4$$
$$U(f, \mathcal{P}') = (15/4)(1/2) + (5)(1/2) + (9)(1) + (47/4)(1/2) + (15)(1/2) = 99/4$$

As we expect from Lemma 5.1.9, since $\mathcal{P}'$ is a refinement of $\mathcal{P}$,

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{P}).$$

'B' Section

1. Recall that on a previous problem set, we showed that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{1}$$

(a) Show that

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0. \tag{2}$$

You may only use the fact in equation (1) above and other general facts about limits. (In other words, no L'Hopital's Rule, which we have not discussed (yet) in this course – see part (e) of question 1 in the 'B' section below.)

Solution: We have

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos(x)}{x} &= \lim_{x \to 0} \frac{1 - \cos(x)}{x} \cdot \frac{1 + \cos(x)}{1 + \cos(x)} \\
&= \lim_{x \to 0} \frac{1 - \cos^2(x)}{x(1 + \cos(x))} \\
&= \lim_{x \to 0} \frac{\sin^2(x)}{x(1 + \cos(x))} \\
&= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{(1 + \cos(x))} \\
&= 1 \cdot 0 = 0
\end{aligned}$$

by the "big theorem" on limits.

(b) Use the limits in (1) and (2) above to show that if $f(x) = \sin(x)$ then f is differentiable at all real c, and $f'(c) = \cos(c)$. (Hint: The best way to do this is to set up the limit for the derivative like this:

$$f'(c) = \lim_{x \to c} \frac{\sin(x) - \sin(c)}{x - c} = \lim_{h \to 0} \frac{\sin(c + h) - \sin(c)}{h}.$$

Solution: We have

$$\begin{aligned}
\lim_{h\to 0}\frac{\sin(c+h)-\sin(c)}{h} &= \lim_{h\to 0}\frac{\sin(c)\cos(h)+\cos(c)\sin(h)-\sin(c)}{h} \\
&= \cos(c)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} - \sin(c)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h} \\
&= \cos(c)\cdot 1 - \sin(c)\cdot 0 \quad \text{(using (1) and (2))} \\
&= \cos(c).
\end{aligned}$$

(c) Similarly, show that if $g(x)=\cos(x)$, then g is differentiable at all real c and $g'(c) = -\sin(c)$.

Solution: We have

$$\begin{aligned}
g'(c) = \lim_{x\to c}\frac{\cos(x)-\cos(c)}{x-c} &= \lim_{h\to 0}\frac{\cos(c+h)-\cos(c)}{h} \\
&= \lim_{h\to 0}\frac{\cos(c)\cos(h)-\sin(c)\sin(h)-\cos(c)}{h} \\
&= -\sin(c)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} + \cos(c)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h} \\
&= -\sin(c)\cdot 1 + \cos(c)\cdot 0 \\
&= -\sin(c).
\end{aligned}$$

2. (Applications and Extensions of the "Most Valuable Theorem")

(a) Prove that if f is differentiable on (a,b) and there exists a positive constant M such that $|f'(x)| \le M$ for all $x\in(a,b)$, then f is uniformly continuous on (a,b).

Solution: By the MVT we have that if $x_1 < x_2$ are any elements of the interval (a,b), then there is some c in the interval (x_1,x_2) such that

$$f(x_2)-f(x_1) = f'(c)(x_2-x_1).$$

Taking absolute values we obtain

$$|f(x_2)-f(x_1)| \le M|x_2-x_1|.$$

This shows that f is Lipschitz continuous on (a,b) as in last week's problem set, and it follows that f is uniformly continuous on (a,b).

(b) Let f be a function such that the first n derivatives $f', f'', \ldots, f^(n)$ exist for all $x\in\mathbf{R}$, and assume that

$$f(x_1) = f(x_2) = \cdots = f(x_{n+1})$$

for distinct $x_1, x_2, \ldots, x_{n+1} \in \mathbf{R}$. Show that $f^n(c) = 0$ for some $c \in \mathbf{R}$. (An induction proof is a possibility here. What is the base case?)

Solution: We prove this by induction on n. The case $n = 1$ follows from Rolle's Theorem applied to f on the interval $[x_1, x_2]$. Since $f(x_1) = f(x_2)$, there is some $c \in (x_1, x_2)$ such that $f'(c) = 0$. Now assume that whenever g is a function whose first k derivatives exist on an interval containing $k+1$ distinct points and $g(x_1) = g(x_2) = \cdots = g(x_{k+1})$, then $g^{(k)}(c) = 0$ for some c in that interval. Now consider a function f whose first $k+1$ derivatives exist on some interval and assume $f(x_1) = f(x_2) = \cdots = f(x_{k+2})$ for distinct $x_1, \ldots, x_{k+2}$ in that interval. We may assume here that $x_1 < x_2 < \cdots < x_{k+2}$ by relabeling if necessary. Apply Rolle's Theorem to f on each of the intervals $[x_i, x_{i+1}]$ for $i = 1, 2, \ldots, k+1$. This gives $c_i \in (x_i, x_{i+1})$ with $f'(c_1) = f'(c_2) = \cdots = f'(c_{k+1}) = 0$. Since those intervals are disjoint, it follows that the c_i must be distinct. Moreover, if the first $k+1$ derivatives of f exist on an interval containing $x_1, \ldots, x_{k+2}$, then the first k derivatives of f' exist on that same interval. Hence we can apply the induction hypothesis to $g = f'$. By the induction hypothesis, there exists some c where the kth derivative of f' is zero. But $(f')^{(k)} = f^{(k+1)}$ by the definition of the higher derivatives of f. Therefore we have shown that there is a c where $f^{(k+1)}(c) = 0$ and we are done by induction.

(c) Let f be differentiable and assume that f' is strictly increasing on $\mathbf{R}$. If $f(a) = f(b)$, where $a < b$, then show that $f(x) < f(a) = f(b)$ for all $a < x < b$.

Solution: Let $a < x < b$. The hypotheses of the MVT are satisfied on both intervals $[a, x]$ and $[x, b]$. Hence there are $c_1 \in (a, x)$ and $c_2 \in (x, b)$ such that $f'(c_1) = \frac{f(x)-f(a)}{x-a}$ and $f'(c_2) = \frac{f(b)-f(x)}{b-x}$. If $f(x) \geq f(b) = f(a)$, then from these consequences of the MVT, it would follow that $f'(c_1) \geq 0$ and $f'(c_2) \leq 0$ But note that $a < c_1 < x < c_2 < b$, so by the assumptions, $f'(c_1) < f'(c_2)$. Hence we cannot have $f(x) \geq f(a) = f(b)$.

(d) Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Show there exists $c \in (a, b)$ such that $g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$. (Hint: Consider the linear combination $h(x) = g(x)(f(b) - f(a)) - f(x)(g(b) - g(a))$.

Solution: h is a linear combination of continuous functions on $[a, b]$, hence is continuous on $[a, b]$. Similarly, it is differentiable on (a, b). We have

$$h'(x) = g'(x)(f(b) - f(a)) - f'(x)(g(b) - g(a)).$$

Note that

$$h(a) = g(a)f(b) - g(a)f(a) - f(a)g(b) + f(a)g(a) = g(a)f(b) - f(a)g(b).$$

Similarly

$$h(b) = g(b)f(b) - g(b)f(a) - f(b)g(b) + f(b)g(a) = g(a)f(b) - f(a)g(b).$$

By the MVT (in the Rolle's Theorem special case), there is a $c \in (a, b)$ where $h'(c) = 0$, and then

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)),$$

which is what we had to prove. (*Comment:* Applying the MVT separately to f and g on the interval $[a, b]$ and getting $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $c \in (a, b)$ and $g'(c) = \frac{g(b)-g(a)}{b-a}$, while tempting, *does not yield a correct proof* because there is no reason the two c's have to be the same(!))

(e) Use part (d) to deduce that if f, g are differentiable on a deleted neighborhood of c with $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, if $g'(x) \neq 0$ on that deleted neighborhood and if

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L, \tag{3}$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L \tag{4}$$

as well. That is, show that the limit on the left side of the equation in (4) exists and equals L. (This is a basic form of L'Hopital's Rule for limits.)

Solution: Since f and g are differentiable and continuous on some open interval $(c - a, c)$ with $a > 0$ and $\lim_{x\to c^-} f(x) = \lim_{x\to c^-} g(x) = 0$, if $f(c)$ and $g(c)$ are not defined directly, we can define $f(c) = g(c) = 0$ to get continuity at c. We want to apply part (d) to f, g on $[c - \delta, c]$, but we must make sure that $g(c - \delta) \neq 0$. If we did have $g(c-\delta) = 0$, then by Rolle's Theorem, $g'(\xi) = 0$ for some ξ between $c-\delta$ and c. Since we assumed that $g'(x)$ was not zero on the whole deleted neighborhood, this cannot happen. Hence for each δ, we get that there is some $\xi \in (c - \delta, c)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(c-\delta) - f(c)}{g(c-\delta) - g(c)} = \frac{f(c-\delta)}{g(c-\delta)}.$$

Now let $\delta \to 0^+$ and use the assumption (3), we get (since $c - \delta < \xi < c$)

$$\lim_{\delta\to 0^+} \frac{f(c-\delta)}{g(c-\delta)} = \lim_{\xi\to c^-} \frac{f'(\xi)}{g'(\xi)} = L.$$

A similar argument "from the other side" shows

$$\lim_{\delta\to 0^+} \frac{f(c+\delta)}{g(c+\delta)} = \lim_{\xi\to c^+} \frac{f'(\xi)}{g'(\xi)} = L.$$

Therefore

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L$$

as required to establish (4). (*Comment:* If you assume more about f, g, there is also a somewhat simpler proof. For instance, if you assume that f', g' exist and are continuous on a whole interval containing c and $g'(x) \neq 0$ on that interval, then you can argue like this

$$\begin{aligned}
\lim_{x\to c} \frac{f'(x)}{g'(x)} &= \frac{f'(c)}{g'(c)} \quad \text{assuming } f', g' \text{ continuous at } c \\
&= \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} \\
&= \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} \\
&= \lim_{x\to c} \frac{f(x)}{g(x)} \qquad \text{since } f(c) = g(c) = 0
\end{aligned}$$

I gave credit for this argument, but you should note that it uses a lot more assumptions than what was given(!)

3. Show that $\int_0^3 f$ exists and determine its value if

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 3 & \text{if } 1 < x \leq 3. \end{cases}$$

(That is, show that f is integrable according to our definition and determine the value of $\int_0^3 f$.)

Solution: Let $\varepsilon > 0$ and consider the partition

$$\mathcal{P}_\varepsilon = \left\{0, 1 - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}, 3\right\}$$

For this partition we have

$$m_1 = 1 = M_1, m_2 = 1 \text{ while } M_2 = 3, m_3 = M_3 = 3.$$

Hence the difference

$$U(f, \mathcal{P}_\varepsilon) - L(f, \mathcal{P}_\varepsilon)$$

is

$$(1)\left(1 - \frac{\varepsilon}{4}\right) + (3)\left(\frac{\varepsilon}{2}\right) + (3)\left(2 - \frac{\varepsilon}{4}\right) - (1)\left(1 - \frac{\varepsilon}{4}\right) - (1)\left(\frac{\varepsilon}{2}\right) - (3)\left(2 - \frac{\varepsilon}{4}\right) = \varepsilon.$$

Since this can be made as small as we like, the function f is integrable and the value of the integral is the limit as $\varepsilon \to 0$ of either the upper or lower sum, namely

$$\int_0^3 f = 7.$$