Mathematics 242 – Principles of Analysis
Solutions for Problem Set 7 – **due:** Friday, April 4

'A' Section

1. Let $f(x) = \frac{15x}{x^4+3x^2+1}$. Use the Intermediate Value and/or Extreme Value Theorems to show the following:

A) For all $k \in [-3, 3]$, there exist $c \in [-1, 1]$ such that $f(c) = k$.

Solution: First, $f(x)$ is a rational function and $x^4 + 3x^2 + 1 \geq 1$ for all $x \in \mathbf{R}$, so it follows that $f(x)$ is continuous at all $c \in \mathbf{R}$. We have $f(-1) = -3$ and $f(1) = 3$. Therefore, by the IVT, for all $k \in [-3, 3]$, there exist $c \in [-1, 1]$ such that $f(c) = k$.

B) For all k with $0 < k < 3$, there exist some $c \in (1, \infty)$ such that $f(c) = k$.

Solution: We see by the "Big Theorem" on function limits that $\lim_{x\to\infty} f(x) = 0$. Hence if we have any k with $0 < k < 3$, then there exists a $B > 1$ such that $f(B) < k < 3 = f(1)$. Applying the IVT on the interval $[1, B]$, we see there is a $c \in (1, B) \subset (1, \infty)$ such that $f(c) = k$.

C) There is a $c \in (0, 1)$ where $f(c) = 3$.

Solution: Consider the function

$$g(x) = f(x) - 3 = \frac{15x - 3x^4 - 9x^2 - 3}{x^4 + 3x^2 + 1}$$

We see that $g(0) = -3 < 0$ and $g(1/2) = \frac{33}{29} > 0$. $g(x)$ is also continuous on $[0, 1/2]$. Therefore, by the IVT, there exists $c \in (0, 1/2) \subset (0, 1)$ where $g(x) = 0$.

D) There is a $d \in (0, 1)$ where $f'(d) = 0$.

Solution: Consider the number c from part c. We have $f(c) = f(1) = 3$, but f is certainly not constant between c and 1. Therefore by the Extreme Value Theorem, there must exist a maximum value $f(d) > 3$ or a minimum value $f(d) < 3$ for f on the interval $[c, 1]$ attained at some $d \in (c, 1)$. By plotting the function f, we can see that in fact the first statement is the one that is true – there is a maximum. We claim that $f'(d) = 0$. First note that

$$\lim_{x\to d^-} \frac{f(x) - f(d)}{x - d} \geq 0$$

since f has a maximum at d. Similarly,

$$\lim_{x\to d^+} \frac{f(x) - f(d)}{x - d} \leq 0$$

Since f is a rational function whose denominator never equals zero, f is differentiable at d, so these one-sided limits must both exist and be equal. Therefore $f'(d) = 0$.

2. Show that there is a solution of the equation $\tan(x) = x$ in the interval $\left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$ for every $k \in \mathbf{Z}$.

Solution: We know that $\tan(x)$ has infinite discontinuities at $\frac{(2k-1)\pi}{2}$ and $\frac{(2k+1)\pi}{2}$ and hence

$$\lim_{x \to \frac{(2k-1)\pi}{2}^+} \tan(x) = -\infty \qquad \lim_{x \to \frac{(2k+1)\pi}{2}^-} \tan(x) = +\infty$$

It follows that we also have

$$\lim_{x \to \frac{(2k-1)\pi}{2}^+} \tan(x) - x = -\infty \qquad \lim_{x \to \frac{(2k+1)\pi}{2}^-} \tan(x) - x = +\infty$$

Therefore, $\tan(x) - x$ must change sign on some interval $[a, b] \subset \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$ and the Intermediate Value Theorem implies $\tan(x) - x = 0$ somewhere in that interval.

3. Using the definition of the derivative, find the value of $f'(c)$, or say why f is not differentiable at $x = c$:

A) $f(x) = x^3 + 2x - 4$ at $c = 2$.

Solution: We have

$$\begin{aligned} f'(2) &= \lim_{x \to 2} \frac{x^3 + 2x - 12}{x - 2} \\ &= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 6)}{x - 2} \\ &= \lim_{x \to 2} x^2 + 2x + 6 \\ &= 14. \end{aligned}$$

B) $f(x) = \sin(|x|)$ at $c = 0$. Hint: Look back at Problem Set 6, B 2.

Solution: $f'(0)$ does not exist for this function because

$$\lim_{x \to 0^+} \frac{\sin(|x|)}{x} = +1$$

by the indicated problem on Problem Set 6, while

$$\lim_{x \to 0^-} \frac{\sin(|x|)}{x} = \lim_{x \to 0^-} \frac{-\sin(x)}{x} = -1.$$

Since the one-sided limits are not equal, the derivative at 0 does not exist.

C) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ 2x - 1 & \text{if } x \leq 1 \end{cases}$$

at $c = 1$.

Solution: We have

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^+} x + 1 = 2.$$

On the other hand,

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{2x - 2}{x - 1} = \lim_{x\to 1^-} 2 = 2.$$

Since the one-sided limits exist and are equal, $f'(1)$ exists and equals 2.

D) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

at $c = 0$.

Solution: We have for $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

Given any $\varepsilon > 0$, if we take $\delta = \varepsilon$, then for all x in the deleted neighborhood defined by $0 < |x| < \varepsilon$,

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| < \varepsilon.$$

It follows that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0 = f'(0).$$

(It is not too hard to show that $f'(c)$ exists only for this one $c = 0$. This function is not differentiable anywhere else.)

'B' Section

1. Let f be continuous on $[0, 1]$ with $f(0) < 0$ and $f(1) > 1$. Suppose that g is another continuous function on $[0, 1]$ such that $g(0) \geq 0$ and $g(1) \leq 1$. Show that there exists some $c \in (0, 1)$ such that $f(x) = g(x)$.

Solution: Let $h(x) = f(x) - g(x)$. Since f, g are continuous on $[0, 1]$, the same is true for h. By the given information, $h(0) = f(0) - g(0) < 0$ and $h(1) = f(1) - g(1) > 0$. Therefore, the IVT implies that $h(c) = 0$ for some $c \in (0, 1)$. But then $0 = h(c) = f(c) - g(c)$, so $f(c) = g(c)$.

2. Let f be continuous on $[a, b]$ with $f(a) < k < f(b)$. Here is a variation on our proof of the Intermediate Value Theorem.

A) Let
$$T = \{x \in [a, b] \mid f(x) > k\}.$$
Show that T has a greatest lower bound and that $f(\text{glb}(T)) = k$.

Solution: T is contained in the interval $[a, b]$, so it is a bounded subset of $\mathbf{R}$. Then $c = \text{glb}(T)$ exists by the LUB Axiom. Note that $a < c$ since $f(a) < k$. Hence the interval $[a, c)$ is contained in the complement of T. If we let $\{x_n\}$ be any sequence contained in $[a, c)$ converging to c, then since f is continuous, $\lim_{n\to\infty} f(x_n) = f(c)$. But $f(x_n) \le k$ for all n, so

$$f(c) = \lim_{n\to\infty} f(x_n) \le k \tag{1}$$

also (by Corollary 2.2.8 in the text). On the other hand, given any $\varepsilon > 0$, $c + \varepsilon$ is not a lower bound for T, so there exists some $x \in T$ such that $c \le x < c + \varepsilon$. Apply this for $\varepsilon = \frac{1}{n}$ for each natural number. Then we get a sequence x_n' such that $x_n' \in T$ for all n and $c \le x_n' < c + \frac{1}{n}$. It follows easily that $x_n' \to c$ as $n \to \infty$. Therefore since f is continuous at c, $\lim_{n\to\infty} f(x_n') = f(c)$. But $x_n' \in T$ for all n, so $f(x_n') > k$. Hence

$$f(c) = \lim_{n\to\infty} f(x_n') \ge k. \tag{2}$$

The two inequalities (1) and (2) show that $f(c) = k$.

B) Will this $\text{glb}(T)$ always be the same as the c we found in our proof of the IVT with $f(c) = k$? If so, prove they are the same; if not, give a counterexample.

Solution: In the proof we did in class we considered
$$S = \{x \in [a, b] \mid f(x) \le k\}$$
and we showed that if $c' = \text{lub}(S)$, then $f(c') = k$. The c found in part A and the c' here do not have to be the same. For instance, let $f(x) = x^3 - 2x + 1$ on $[-2, 2]$. We have $f(-2) = -3$ and $f(2) = 5$. So the IVT will apply for any k with $-3 < k < 5$. Consider $k = 0$. The equation $x^3 - 2x + 1 = 0$ actually has three different roots in the interval $[-2, 2]$: One between -2 and -1 (call this one α), a second between $1/2$ and 1 (call this one β), and a third at $x = 1$. The set T as in part A is the union $T = (\alpha, \beta) \cup (1, 2)$, so $c = \text{glb}(T) = \alpha$. On the other hand, the set S as in the proof we did in class is $S = [-2, \alpha] \cup [\beta, 1]$, so $c' = \text{lub}(S) = 1$.

3. This property deals with another property of real-valued functions of a real variable sometimes called *Lipschitz continuity.*

A) Let f be a function on an interval I with the property that there exists a strictly positive constant k such that $|f(x) - f(x')| \le k|x - x'|$ for all $x, x' \in I$ (this is the definition of Lipschitz continuity). Show that f is uniformly continuous on I.

Solution: Given $\varepsilon > 0$, let $\delta = \varepsilon/k$. Then for any $x, x' \in I$ such that $|x - x'| < \delta = \varepsilon/k$, it follows that

$$|f(x) - f(x')| \leq k|x - x'| < k \cdot \varepsilon/k = \varepsilon.$$

This shows that the definition of uniform continuity is satisfied for f on I.

B) The converse of the statement in part A is not true: Show that $f(x) = x^{1/3}$ is uniformly continuous on $[-1, 1]$, but there is no constant k such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in [-1, 1]$. Hint: Think slopes of secant lines to the graph $y = x^{1/3}$.

Solution: First, $f(x)$ is continuous on $[-1, 1]$, hence it is uniformly continuous by the result of Theorem 3.6.8 (proved in class before Easter break). Let $x' = 0$ and take arbitrary $x > 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^{1/3}}{x} = \frac{1}{x^{\frac{2}{3}}}.$$

But $\lim_{x \to 0^+} \frac{1}{x^{\frac{2}{3}}} = +\infty$. In other words, the value of the difference quotient will get unboundedly large as $x \to 0^+$. Hence there is no single k such that

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \leq k$$

for all x in $[-1, 1]$. But that shows that there is no k such that $|f(x) - f(0)| \leq k|x - 0|$ for all x in $[-1, 1]$.