Mathematics 242 – Principles of Analysis
Solutions for Problem Set 6 – **due:** Monday, March 24

'A' Section

1. Determine whether each of the following limits exists using the "big theorem" for function limits and other results from section 3.2 of the text as needed.

(a) $\lim_{x\to 1} x^2 - 5x + 3$

Solution: By parts 1 and 2 of the "big theorem," the limit is

$$\lim_{x\to 1} x^2 - 5\lim_{x\to 1} x + 3 = 1 - 5 + 1 = -1.$$

(b) $\lim_{x\to \frac{1}{3}} x + \frac{1}{x^2}$

Solution: By parts 1 and 3 of the "big theorem," the limit is $1/3 + 9 = \frac{28}{3}$.

(c) $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

Solution: The limit is $\frac{3}{2}$. Proof: For $x \neq 1$, we see

$$\frac{x^3-1}{x^2-1} = \frac{(x^2+x+1)(x-1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

This shows $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$.

(d) Let

$$f(x) = \begin{cases} x^{1/3}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$

and consider $\lim_{x\to 0} f(x)$.

Solution: The limit is 0 by the limit squeeze theorem. We have $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \neq 0$. So

$$-x^{1/3} \le f(x) \le x^{1/3}$$

for all $x \neq 0$. Since $\lim_{x\to 0} -x^{1/3} = \lim_{x\to 0} x^{1/3} = 0$, $\lim_{x\to 0} f(x) = 0$ also.

2. Which of the functions in question 1 are *continuous* at the indicated c in the limits there? Explain.

Solution: The functions in parts (a) and (b) of problem 1 are continuous at the given c since $\lim_{x\to c} f(x) = f(c)$. The function in part (c) is not continuous at 2 since $f(1)$ is not defined.

The function in part (d) is not continuous at $x = 0$ since $\lim_{x\to 0} f(x) = 0 \neq 3 = f(0)$ (by the definition of $f(x)$).

3. True-False. For the true statements, give a short proof. For the false statements give a counterexample.

(a) If $\lim_{x\to 1} f(x) = e - \frac{28}{10}$, then there exists a $\delta > 0$ such that $f(x) < 0$ for all x with $0 < |x - 1| < \delta$.

This is TRUE. The reason is that $a = e - \frac{28}{10} < 0$ (since $e \doteq 2.71828 < 2.8$). So if we take $\varepsilon = |a|/2$, then there is a corresponding $\delta > 0$ such that for x with $0 < |x-1| < \delta$,

$$|f(x) - a| < |a|/2.$$

But this implies $f(x) < a + |a|/2 = a/2 < 0$ for all such x.

(b) If $|f(x)| \leq x^3$ for all x and $\lim_{x\to 2} f(x)$ exists, then $\lim_{x\to 2} f(x) \leq 8$.

Solution: This is TRUE. We have $f(x) \leq |f(x)| \leq x^3$ so $\lim_{x\to 2} f(x) \leq \lim_{x\to 2} x^3 = 8$, using Theorem 3.2.8.

(c) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by this rule:

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ -2x & \text{if } x \text{ is irrational.} \end{cases}$$

Then $\lim_{x\to 0} f(x)$ exists and equals 0.

Solution: This is TRUE. Given any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then for all x with $0 < |x - 0| < \delta = \varepsilon/2$, we have $|2x| = |-2x| = 2|x|$ for rational and irrational x, so

$$|f(x)| = 2|x| < 2\frac{\varepsilon}{2} = \varepsilon.$$

This shows the limit is 0 as claimed.

(d) If $f(x) < g(x)$ on a deleted neighborhood of c, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then $L < M$.

Solution: This is FALSE. Counterexample: Let $f(x) = x^4$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all x with $0 < |x - 0| < 1$. But $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. (The statement would be true if it said $L \leq M$.)

'B' Section

1. Show that your answers for parts a and d of 1 on the A section are correct using the ε, δ definition (*not* the "big theorem" or other results from Chapter 3, section 1 of the text.)

Solution: Part (a) first. Given $\varepsilon > 0$, let $\delta = \min(1, \varepsilon/4)$. For all x with $0 < |x-1| < \delta < 1$, we have $0 < x < 2$, so $|x-4| < 4$ and hence

$$\begin{aligned} |x^2 - 5x + 3 - (-1)| &= |x^2 - 5x + 4| \\ &= |x-4||x-1| \\ &< 4 \cdot \frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

This shows $\lim_{x\to 1} x^2 - 5x + 3 = -1$.

Part (b). Given $\varepsilon > 0$, let $\delta = \varepsilon^3 > 0$. Then For all x with $0 < |x - 0| < \delta = \varepsilon^3$, we have

$$|f(x) - 0| = \left| x^{1/3} \sin\left(\frac{1}{x}\right)\right| = |x|^{1/3} \left|\sin\left(\frac{1}{x}\right)\right| \le |x|^{1/3} < (\varepsilon^3)^{1/3} = \varepsilon.$$

This shows $\lim_{x\to 0} f(x) = 0$.

2. Assume that $\lim_{x\to c} f(x) = L$.

(a) Show that there exists a constant B and $\delta > 0$ such that $|f(x)| \le B$ for all x in the deleted neighborhood $\{x \in \mathbf{R} \mid 0 < |x - c| < \delta\}$.

Solution: Since $\lim_{x\to c} f(x) = L$, letting $\varepsilon = 1$, there is a corresponding $\delta > 0$ such that $|f(x) - L| < 1$ for all x in the deleted neighborhood defined by $0 < |x - c| < \delta$. But for those x, $L - 1 < f(x) < L + 1$, so $|f(x)| \le \max(|L + 1|, |L - 1|)$. We can take $B = \max(|L + 1|, |L - 1|)$.

(b) Using part (a), *not* the limit product rule, show that $\lim_{x\to c}(f(x))^n = L^n$ for all integers $n \ge 1$.

Solution: Let B and δ_0 be as in part (a). That is assume that $|f(x)| \le B$ for all x with $0 < |x - c| < \delta_0$. Given ε, since $\lim_{x\to c} f(x) = L$, we have $|f(x) - L| < \varepsilon/M$, where

$$M = B^{n-1} + B^{n-2}|L| + \cdots + B|L|^{n-2} + |L|^{n-1}$$

for all x with $0 < |x - c| < \delta_1$ for some $\delta_1 > 0$. Let $\delta = \min(\delta_0, \delta_1)$. Then for all x with $0 < |x - c| < \delta$, we have (using the triangle inequality on the second factor on the right side):

$$\begin{aligned} |(f(x))^n - L^n| &= |f(x) - L||(f(x))^{n-1} + (f(x))^{n-2}L + \cdots + f(x)L^{n-2} + L^{n-1}| \\ &\le |f(x) - L|(|f(x)|^{n-1} + |f(x)|^{n-2}|L| + \cdots + |f(x)||L|^{n-2} + |L|^{n-1}) \\ &< \frac{\varepsilon}{M}(B^{n-1} + B^{n-2}|L| + \cdots + B|L|^{n-2} + |L|^{n-1}) \\ &= \frac{\varepsilon}{M} \cdot M = \varepsilon. \end{aligned}$$

This shows $\lim_{x\to c}(f(x))^n = L^n$.

(c) Assume that $f(x) \geq 0$ on some deleted neighborhood of $x = c$. Show that

$$\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}.$$

(*Hint:* It may help to treat the cases $L = 0$ and $L \neq 0$ separately.)

Solution: First suppose $L = 0$. Then for all $\varepsilon > 0$, there exist corresponding $\delta > 0$ such that $|f(x)| < \varepsilon^2$ for all x with $0 < |x - c| < \delta$. But then for the same x, we have $|\sqrt{f(x)}| < \varepsilon$. So $\lim_{x\to c} \sqrt{f(x)} = 0 = \sqrt{0}$. Now assume $L \neq 0$ (so $L > 0$). Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon\sqrt{L}$ for all x with $0 < |x - c| < \delta$. For these x,

$$\begin{aligned} |\sqrt{f(x)} - \sqrt{L}| &= \frac{|f(x) - L|}{f(x) + \sqrt{L}} \\ &\leq \frac{|f(x) - L|}{\sqrt{L}} \\ &< \frac{\varepsilon\sqrt{L}}{\sqrt{L}} \\ &= \varepsilon. \end{aligned}$$

This shows $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

3. In this problem you will show that

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

For $0 < \theta < \frac{\pi}{2}$, the point $P = (\cos(\theta), \sin(\theta)) = (x, y)$ lies on the arc of the unit circle $x^2 + y^2 = 1$ in the first quadrant.

(a) Let $O = (0, 0)$, $Q = (\cos(\theta), 0)$, and $R = (1, 0)$. (Draw a picture!) By considering the areas of the triangle ΔOQP and the circular sector ORP, deduce that if $0 < \theta < \frac{\pi}{2}$, then $\sin(\theta)\cos(\theta) \leq \theta$. (You may use "intuitively reasonable" facts about areas such as the statement that if one plane region $\mathcal{R}$ is completely completely contained in a second region $\mathcal{S}$, then area$(\mathcal{R}) \leq$ area$(\mathcal{S})$.)

Solution: The area of the triangle ΔOQP is $\frac{1}{2}\sin(\theta)\cos(\theta)$. The area of the sector is $\frac{1}{2}\theta$, since the area of the circular sector with angle Θ of a circle of radius r is $\frac{\Theta r^2}{2}$. Since the sector completely contains the triangle, the desired inequality follows.

(b) Now take the tangent line to the circle at R (a vertical line), and let $S = (1, \tan(\theta))$ be the intersection of that line and the radius OP (extended). Considering the areas of the triangle ΔORS and the sector ORP as above, explain why $\theta \leq \tan(\theta)$.

Solution: The area of the triangle ΔORS is $\frac{1}{2}\tan(\theta)$. (By the way, in case you never have seen this before, this is the reason that the tangent function is known by that name!) This time the sector ORP is completely contained in the triangle, so the inequality follows again.

(c) Combine parts (a) and (b) to deduce that if $0 < \theta < \frac{\pi}{2}$, then

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}.$$

Solution: Combining parts (a) and (b), we see

$$\sin(\theta)\cos(\theta) \leq \theta \leq \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

Since $\sin(\theta) > 0$ for the θ in this range, it follows that

$$\cos(\theta) \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

and the desired inequalities follow by taking reciprocals.

(d) Using the one-sided form of Theorem 3.2.9 (The Limit Squeeze Theorem), show that

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1.$$

(You will need to use the fact that $\cos(\theta)$ is continuous at $\theta = 0$.)

Solution: Since $\cos(\theta)$ is continuous at $\theta = 0$, we have $\lim_{\theta\to 0^+}\cos(\theta) = 1$ and hence $\lim_{\theta\to 0^+}\frac{1}{\cos(\theta)} = 1$ as well. By the one-sided version of the Limit Squeeze Theorem,

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$$

also.

(e) Now, for $-\frac{\pi}{2} < \theta < 0$, show that $\frac{\sin(\theta)}{\theta} = \frac{\sin(|\theta|)}{|\theta|}$ and use this to see that

$$\lim_{\theta\to 0^-} \frac{\sin(\theta)}{\theta} = 1$$

as well.

Solution: This follows from the fact that sin is an odd function. If $\theta < 0$, then

$$\frac{\sin(|\theta|)}{|\theta|} = \frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta}.$$

By letting $\varphi = |\theta| > 0$, from part (d) we see that

$$\lim_{\theta \to 0^{-1}} \frac{\sin(\theta)}{\theta} = \lim_{\theta \to 0^{-}} \frac{\sin(|\theta|)}{|\theta|} = \lim_{\varphi \to 0^{+}} \frac{\sin(\varphi)}{\varphi} = 1.$$

(f) Finally, explain how parts (d) and (e) combine to show the statement at the start of the problem.

Solution: The desired statement follows from parts (d) and (e) and Theorem 3.3.4 (equality of the two one-sided limits implies the two-sided limit exists and equals the common value).