MATH 242 – Principles of Analysis
Solutions for Problem Set 2 – due: Feb. 8

'A' Section

1. Let $x \in [-1, 2]$. Determine the largest and smallest values of $|x - 5|$, $|x + 5|$.
   Solution: If $x \in [-1, 2]$, then $x - 5 \in [-6, -3]$, so the largest and smallest values of $|x - 5|$ are 6 and 3 respectively. (You can also see these as the distances from -1 to 5 and 2 to 5 along the number line, thinking geometrically.) Next, $x + 5 \in [4, 7]$, so the largest and smallest values of $|x + 5|$ are 7 and 4 respectively.
2. Use the binomial theorem (Theorem 1.4.1) for all parts of this problem.
   a. Expand using the binomial theorem and simplify as much as possible:
   $$(a^2 + 3b)^5.$$
   Solution: We have
   $$(a^2 + 3b)^5 = a^{10} + 15a^8b + 90a^6b^2 + 135a^4b^3 + 405a^2b^4 + 243b^5.$$
   b. What is the coefficient of x^3 in the expansion of
   $$\left(\frac{x^5 + 3}{x^2}\right)^3.$$
   Solution: This coefficient is zero. The powers of x actually appearing in the binomial expansion will be $(x^5)^3 = x^{15}$, $(x^5)^2x^{-2} = x^8$, $x^5(x^{-2})^2 = x$ and $(x^{-2})^3 = x^{-6}$.
   c. What is $\sum_{k=0}^{n} \binom{n}{k} 2^k$? Explain.
   Solution: From the binomial theorem this sum is what we obtain from
   $$(1 + 2)^n = 3^n.$$
   d. What is $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$? Explain.
   Solution: From the binomial theorem this sum is what we obtain from
   $$(1 - 1)^n = 0^n = 0.$$
3. For each of the following statements, say whether the statement is true or false. If it is false, give a counterexample; if it is true, give a short reason.

a. A set $A \subset \mathbf{R}$ is bounded if there exists some $B > 0$ such that $|x| \leq B$ for all $x \in A$.

Solution: TRUE – the set is bounded above by B and below by $-B$.

b. If $A, B \subset \mathbf{R}$ are bounded, then $A \cap B$ is also bounded.

Solution: TRUE – If A is bounded above by M_A and B is bounded above by M_B, then $A \cap B$ is bounded above by $\min(M_A, M_B)$. Similarly, if A is bounded below by m_A and B is bounded below by m_B, then $A \cup B$ is bounded above by $\max(m_A, m_B)$.

c. If $A, B \subset \mathbf{R}$ are bounded, then $D = \{x + y \mid x \in A, y \in B\}$ is also bounded.

Solution: TRUE – Suppose A is bounded above by M_A and B is bounded above by M_B. Similarly, suppose A is bounded below by m_A and B is bounded below by m_B. Then for all $x \in A$ and $y \in B$ we have $m_A \leq x \leq M_A$ and $m_B \leq y \leq M_B$. It follows that $m_A + m_B \leq x + y \leq M_A + M_B$. Therefore D is bounded.

d. If $A, B \subset \mathbf{R}_{>0}$ are bounded, then $Q = \{y/x \mid x \in A, y \in B\}$ is also bounded.

Solution: FALSE – Let $A = (0, 1)$ and let $B = \{1\}$. The set Q is the set $\{1/y \mid y \in (0, 1)\}$ which is not bounded above.

4.

a. Let $A = [0, 3) \cap (2, 5]$. What is $a = \text{lub } A$? What is $b = \text{glb } A$? Are $a, b \in A$?

Solution: We have $A = (2, 3)$. Therefore $a = 3$, which is not in A. Similarly, $b = 2$ is not in A either.

b. Let $B = \{x \in \mathbf{R} \mid 0 \leq x^2 - 2x + 1 \leq 1\}$. What is $a = \text{lub } B$? What is $b = \text{glb } B$? Are $a, b \in B$?

Solution: B is the set where $(x - 1)^2 \leq 1$, or $x \in [0, 2]$, so $a = 2$ and $b = 0$ are both in B.

'B' Section

1. Let x, y be any real numbers.

a. Show that $|x| - |y| \leq |x - y|$ and deduce that $||x| - |y|| \leq |x - y|$.

Solution: From the usual triangle inequality,

$$|x| = |(x - y) + y| \leq |x - y| + |y|.$$

Subtracting, we obtain $|x| - |y| \leq |x - y|$ as desired. Similarly, reversing the roles of x, y, we have $|y| - |x| \leq |y - x| = |x - y|$. Since either $|x| \geq |y|$ or $|y| \geq |x|$ is true, we have either $||x| - |y|| = |x| - |y|$ or $||x| - |y|| = |y| - |x|$. Since both of those are $\leq |x - y|$, it follows that $||x| - |y|| \leq |x - y|$ as desired.

b. Show that if $x, y > 0$, then $x < y$ is equivalent to $x^n < y^n$ for all $n \geq 1$

Solution: $\Rightarrow$: We argue by induction on n. The base case is the same as the hypothesis so there is nothing to prove. Assume $x^k < y^k$ for some positive integer k. Then since $x > 0$, we can multiply by x on both sides to get $x^{k+1} < y^k x$. Similarly, we can multiply the base case $x < y$ by $y^k > 0$ on both sides to get $xy^k < y^{k+1}$. But then transitivity of the order relation implies $x^{k+1} < y^{k+1}$. This shows that $x^n < y^n$ for all $n \geq 1$ by induction. Conversely, if $x^n < y^n$ for all $n \geq 1$, then $x < y$ directly from the case $n = 1$ (!)

c. Show that if $0 < x < y$, then $\sqrt{y} - \sqrt{x} < \sqrt{y - x}$.

Solution: By part b with $n = 2$, since $\sqrt{y} - \sqrt{x} > 0$ and $\sqrt{y - x} > 0$, it suffices to show that

$$(\sqrt{y} - \sqrt{x})^2 < (\sqrt{y - x})^2.$$

But the left side here is $y - 2\sqrt{y}\sqrt{x} + x$ and the right side is $y - x$. We have

$$(y - x) - (y - 2\sqrt{y}\sqrt{x} + x) = 2\sqrt{y}\sqrt{x} - 2x = 2\sqrt{x}(\sqrt{y} - \sqrt{x}).$$

This is > 0 because of the assumption $y > x$ and part b. Hence the desired inequality follows.

2. Let a, b be any real numbers. Define $\max(a, b)$ and $\min(a, b)$ to be the larger and smaller of the two numbers, respectively. (That is, $\max(a, b) = a$ if $a \geq b$ and $\max(a, b) = b$ if $b \geq a$. Similarly for the minimum.) Show that

$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}$$

and

$$\min(a, b) = \frac{a + b}{2} - \frac{|a - b|}{2}.$$

Solution: Geometrically, the average $\frac{a+b}{2}$ is the midpoint of the line segment along $\mathbf{R}$ between a and b and $|a - b|$ is the distance between a and b (the length of the line segment). So starting at the midpoint and going 1/2 the distance to the right gives the maximum of the endpoints, and going 1/2 the distance to the left gives the minimum of the endpoints. More analytically, we can also prove these by breaking into cases. Suppose first that $a \geq b$ so a is the right endpoint. Then $|a - b| = a - b$, and

$$\frac{a + b}{2} + \frac{|a - b|}{2} = \frac{a + b}{2} + \frac{a - b}{2} = a = \max(a, b),$$

while

$$\frac{a + b}{2} + \frac{|a - b|}{2} = \frac{a + b}{2} - \frac{a - b}{2} = b = \min(a, b).$$

If b is the maximum and a is the minimum, then

$$\frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{b-a}{2} = b = \max(a,b),$$

while

$$\frac{a+b}{2} - \frac{|a-b|}{2} = \frac{a+b}{2} - \frac{b-a}{2} = a = \min(a,b).$$

3. Show by mathematical induction that

$$(1+x)^n \geq 1 + nx$$

for all $x > -1$ and all $n \geq 0$.

Solution: With $n = 0$, we have $1 = 1$, so the base case is established. Now assume the inequality has been proved for $n = k$ and consider the next case $n = k+1$. We have by the induction hypothesis

$$(1+x)^k \geq 1 + kx.$$

Since $x > -1$, $1 + x > 0$, so multiplying this factor on both sides we get

$$(1+x)^{k+1} \geq (1+kx)(1+x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x,$$

since $k \geq 0$ and $x^2 \geq 0$. This establishes the desired inequality in all cases.

4. Find a suitable n_0 and then show by mathematical induction that $n! \geq 5^n$ for all $n \geq n_0$.

Solution: The smallest n_0 such that $n_0! \geq 5^{n_0}$ is $n_0 = 12$ since $12! = 479,001,600$, while $5^{12} = 244,1140,625$. This gives the base case here: $12! > 5^{12}$ is true. For the induction step, assume $k! > 5^k$ and consider $(k+1)!$. We see, by the induction hypothesis (and since $k+1 > 5$ whenever $k \geq 12$):

$$(k+1)! = (k+1)k! > (k+1)5^k > 5 \cdot 5^k = 5^{k+1}.$$