MATH 242 – Principles of Analysis
Solutions for Problem Set 1 – due: Jan. 31

'A' Section

1. Assume that A, B are sets of integers.
   a. What is the contrapositive of the statement: "If x is even then $x \in A \cup B$"? Express without using *not*.
   Solution: The contrapositive of "if p then q" is "if not q then not p." Here, by the DeMorgan Law, $x \notin A \cup B$ is equivalent to $x \in A^c$ and $x \in B^c$. Also, "not even" is equivalent to "odd." So, without using *not*, we can state the contrapositive as "If $x \in A^c$ and $x \in B^c$, then x is odd."
   b. What is the converse of the statement in part a?
   Solution: The converse is: "If $x \in A \cup B$, then x is even."
2. Let $A = \{x \in \mathbf{R} \mid x^2 - 5x + 4 = 0\}$, $B = (0, 1) = \{x \in \mathbf{R} \mid 0 < x < 1\}$ and $C = \{\frac{x}{x^2+9} \mid x \in \mathbf{R}\}$ (Note: C is the range of the function f defined by $f(x) = \frac{x}{x^2+9}$.)
   a. Express the set C as a union of one or more closed intervals $[a, b]$ in $\mathbf{R}$. (Note: You should use facts from calculus to solve this. Don't worry that we have not justified them yet.)
   Solution: The function $f(x) = \frac{x}{x^2+9}$ has $f'(x) = \frac{9-x^2}{(x^2+9)^2}$. This is $= 0$ at $x = \pm 3$. Moreover $f'(x) < 0$ for $x < -3$, $f'(x) > 0$ for $-3 < x < 3$ and $f'(x) < 0$ for $x > 3$. Therefore, at $x = -3$, f has a local minimum with $f(-3) = -1/6$. Similarly, at $x = 3$, f has a local maximum with $f(3) = 1/6$. We also see $\lim_{x\to\pm\infty} f(x) = 0$. Hence $f(-3) = -1/6$ is also an absolute minimum, and $f(3) = 1/6$ is also an absolute maximum. We will prove a general theorem later in the course that shows that every y with $-1/6 < y < 1/6$ must also be in the range, but this can also be checked directly here since the equation

$$y = \frac{x}{x^2 + 9}$$

   can be rearranged to $yx^2 - x + 9y = 0$. If $y = 0$, then $x = 0$. Otherwise, by the quadratic formula this has roots

$$x = \frac{1 \pm \sqrt{1 - 36y^2}}{2y}.$$

   The expression in the square root is nonnegative exactly when $-1/6 \le y \le 1/6$ and we get x with $f(x) = y$ (two of them in fact for $y \ne 0, -1/6, 1/6$). Hence $C = [-1/6, 1/6]$.

b. Find the sets $B \cap A$ and $B \cap C$.

Solution: Since $A = \{1, 4\}$, we see that $B \cap A = \emptyset$ and $B \cap C = (0, 1/6]$.

c. Find the sets $B \cup A$ and $B \cup C$ and express using set notation.

Solution: We have $B \cup A = (0, 1] \cup \{4\} = B$. Then by part a, $B \cup C = (0, 1) \cup [-1/6, 1/6] = [-1/6, 1)$.

3. For n a general natural number, let $B_n = \{0, 2n\}$. What are $\cap_{n=1}^{\infty} B_n$ and $\cup_{n=1}^{\infty} B_n$?

Solution: The union, $\cup_{n=1}^{\infty} B_n$, is the set

$$\{0, 2, 4, \cdots\} = \{2n \mid n \geq 0\},$$

or the set of nonnegative even integers. The intersection, $\cap_{n=1}^{\infty} B_n$, is the set $\{0\}$, since that is the only element in B_n for all $n \geq 1$.

4. Let $I_n = [-1/n, 1/n]$ for any $n \geq 1$. What are $\cap_{n=1}^{\infty} I_n$ and $\cup_{n=1}^{\infty} I_n$. (Explain your reasoning intuitively.)

Solution: Note first that $I_m \subset I_n$ whenever $m > n$. This shows that the union is the same as $I_1 = [-1, 1]$. The intersection contains only 0. We will see in about a week how to justify the claim that for any real $a > 0$, there is some $n \geq 1$ such that $1/n < a$. Hence a is not in the intersection. The same is true on the negative side: for any $b < 0$, there exists some $n \geq 1$ such that $b < -1/n$. Hence b is not in the intersection either. This leaves only 0 which does satisfy $-1/n < 0 < 1/n$ for all $n \geq 1$.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = \tan^{-1}(x)$.

a. Is f one-to-one? Why or why not?

Solution: Yes, the inverse tangent of x is defined as the unique angle θ in the interval $(-\pi/2, \pi/2)$ such that $\tan(\theta) = x$. So $\tan^{-1}(x) = \theta = \tan^{-1}(x')$ implies that $x = x'$.

b. Is f onto? Why or why not?

Solution: No, since the range is just the interval $(-\pi/2, \pi/2)$.

c. If $I = (0, \sqrt{3})$, what is the set $f(I)$? Explain.

Solution: $f(I) = (0, \pi/3)$ since $\tan(0) = 0$ and $\tan(\pi/3) = \sqrt{3}$.

d. If $J = (-\pi/4, \pi/4)$, what is the set $f^{-1}(J)$. Explain.

Solution: $f^{-1}(J) = \{x \mid -\pi/4 < \tan^{-1}(x) < \pi/4\}$, which is the same as $\tan(-\pi/4) < x < \tan(\pi/4)$, so $-1 < x < 1$. Hence $f^{-1}(J)$ is the open interval $(-1, 1)$.

'B' Section

1. Prove part (f) of Theorem 1.1.3 in the text. These are the *De Morgan Laws* for complements.

   Solution: We show $(A \cap B)^c = A^c \cup B^c$. Let $x \in (A \cap B)^c$, then $x \notin A \cap B$, which says $x \notin A$ *or* $x \notin B$. But then $x \in A^c \cup B^c$, and it follows that $(A \cap B)^c \subset A^c \cup B^c$. Conversely, if $x \in A^c \cup B^c$, then $x \notin A$ or $x \notin B$. This shows $x \notin A \cap B$, so $x \in (A \cap B)^c$, and it follows that $A^c \cup B^c \subset (A \cap B)^c$. Since we have both inclusions, $(A \cap B)^c = A^c \cup B^c$. The second statement $(A \cup B)^c = A^c \cap B^c$ is proved similarly.

2. Let A and B be arbitrary sets. Does $A = A - (B - B)$, as we might expect if we looked at the formula through the lens of ordinary algebra? If this is always true, prove it; if it is not, give both a counterexample (an example where the formula is not true), and a correct statement with proof.

   Solution: This is true since for any set B, we have $B - B = \emptyset$. This follows from part g of Theorem 1.1.3, for instance: $B - B = B \cap B^c = \emptyset$. But then $A - \emptyset = A$, since $A - \emptyset = A \cap \emptyset^c = A \cap U = A$ (where U denotes the universal set).

3. Let $f : A \to B$ be a function.

   a. Let C, D be subsets of A. Is it always true that $f(C \cap D) = f(C) \cap f(D)$? If this is always true prove it; if it is not, give a counterexample.

      Solution: This is not true. For instance, let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Let $C = (-1, 0)$ and $D = (0, 1)$. Then $f(C) = f(D) = (0, 1)$, so $f(C) \cap f(D) = (0, 1)$. But $C \cap D = \emptyset$, so $f(C \cap D) = \emptyset$ as well. Note that other similar examples can be constructed any time that f is *not one-to-one*.

   b. Show that f is one-to-one if and only if $f^{-1}(f(C)) = C$ for all subsets C of A.

      Solution: First note that $C \subseteq f^{-1}(f(C))$ for all f and all C since if $x \in C$, then $f(x) \in f(C)$, so $x \in f^{-1}(f(C))$ and hence $C \subseteq f^{-1}(f(C))$. So what we need to show here can be restated as follows: (1) if f is one-to-one, then we need to show $f^{-1}(f(C)) \subseteq C$ for all C. And conversely (2) if $f^{-1}(f(C)) \subseteq C$ for all C, then we need to show that f is one-to-one.

      To prove (1), let f be one-to-one. For each $y \in f(C)$, there is some $x \in C$ such that $f(x) = y$. But if $f(x') = y$, then f being one-to-one implies that $x = x'$. Hence the only elements of A that map to $f(C)$ are the elements of C, so $f^{-1}(f(C)) \subseteq C$.

      To prove (2), let $f^{-1}(f(C)) \subseteq C$ for all subsets C of A. In particular, let $C = \{x\}$ for some particular element $x \in A$. Suppose that $f(x') = f(x)$. Then by definition, x and x' are both elements of $f^{-1}(f(C))$. But by assumption $f^{-1}(f(C)) = \{x\}$, so $x = x'$. This shows that f must be one-to-one.

4. Let $f : A \to B$ and $g : B \to C$.

   a. Show that if f and g are both onto, then $g \circ f : A \to C$ is also onto.

      Solution: Let $z \in C$. Since g is onto, there exists $y \in B$ such that $g(y) = z$. But then since f is onto, there exists $x \in A$ such that $f(x) = y$. Combining these statements, we see that $g(f(x)) = (g \circ f)(x) = z$. Since z was arbitrary, this shows that $g \circ f$ is onto.

   b. Is the converse of the statement in part a true? That is, if you know that $g \circ f$ is onto, does it follow that f and g are onto? Prove or find a counterexample.

      Solution: This statement is *not true*. Let $A = B = \mathbf{R}$ and $C = [0, \infty)$, and let $f : A \to B$ be defined by $f(x) = x^2$ and $g(y) = \sqrt{|y|}$. Then for all $z \in C$ we have $z = g(f(z))$, so $g \circ f$ is onto. However, f is not onto since its range contains no negative numbers. (It does follow in general that g must be onto, but as in the counterexample, if g is not one-to-one, the range of f only needs to contain one inverse image of each element $z \in C$.)