Mathematics 242 – Principles of Analysis
Solutions for Midterm Exam 3
May 2, 2014

I. (15) Let

$$f(x) = \begin{cases} x^{4/5}\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous at $x = 0$? Is f differentiable at $x = 0$? Give complete reasons for your assertions.

Solution: Since $|\sin(1/x)| \leq 1$ for all $x \neq 0$, we have

$$-x^{4/5} \leq f(x) \leq x^{4/5}$$

for all $x \neq 0$. Also $\lim_{x\to 0} \pm x^{4/5} = 0$. By the limit squeeze theorem, $\lim_{x\to 0} f(x) = 0 = f(0)$. Therefore, f is continuous at $x = 0$. However, f is not differentiable at $x = 0$ because

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sin(1/x)}{x^{1/5}}$$

does not exist. For instance, at $x_n = \frac{2}{(4n+1)\pi}$ (a sequence converging to 0), we have

$$\frac{\sin(1/x_n)}{x_n^{1/5}} = \sin((4n+1)\pi/2)\left(\frac{(4n+1)\pi}{2}\right)^{1/5} = \left(2n\pi + \frac{\pi}{2}\right)^{1/5} \to +\infty$$

as $n \to \infty$.

II. Both parts of this question refer to the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 - x^2$.

A) (20) Consider the regular partitions $\mathcal{P}_n$ of the interval $[1, 3]$ and show directly, using the upper and lower sums, that f is integrable on $[1, 3]$.

Solution: Note that f is *decreasing* on $[1, 3]$ since $f'(x) = -2x < 0$ for all x with $1 \leq x \leq 3$. This means the desired statement can either be shown by following the proof of our theorem that monotone functions are integrable, or directly. Here is the direct way:

The partition is

$$\mathcal{P}_n = \{1, 1 + 2/n, 1 + 4/n, \ldots, 3\},$$

with $x_i = 1 + 2i/n$ for $i = 0, 1, \ldots, n$. Hence, since f is smallest at the right endpoint in each subinterval,

$$\begin{aligned} L_{\mathcal{P}_n}(f) &= \sum_{i=1}^{n} \left(1 - (1 + 2i/n)^2\right) \frac{2}{n} \\ &= -\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i \\ &= -\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2} \cdot \frac{n(n+1)}{2} \\ &= -\frac{20}{3} - \frac{8}{n} - \frac{4}{3n^2}. \end{aligned}$$

Similarly, f is largest at the left endpoint in each subinterval, so

$$\begin{aligned} U_{\mathcal{P}_n}(f) &= \sum_{i=1}^{n} \left(1 - (1 + 2(i-1)/n)^2\right) \frac{2}{n} \\ &= -\frac{20}{3} + \frac{8}{n} - \frac{4}{3n^2}. \end{aligned}$$

Therefore, for any given $\varepsilon > 0$, if $n > 16/\varepsilon$,

$$U_{\mathcal{P}_n}(f) - L_{\mathcal{P}_n}(f) = \frac{16}{n} < \varepsilon.$$

This shows that f is integrable.

B) (15) Explain why the hypothesis of the Mean Value Theorem is satisfied for f on the interval $[1, 3]$ and find the number c mentioned in the conclusion.
Solution: f is a polynomial function, so it is differentiable, hence continuous everywhere. On the interval $[1, 3]$, $f(3) - f(1) = -8 - 0 = -8$. The MVT says that there is some $c \in (1, 3)$ where $-8 = f'(c) \cdot (3 - 1)$. Since $-8 = 2f'(c) = -4c$, this is true for $c = 2$.

III. (20) Show that if f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Solution: By a previous result we know that $f(x)$ continuous on $[a, b]$ implies that $f(x)$ is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon/(b - a)$ whenever $|x - x'| < \delta$ (with $x, x' \in [a, b]$, of course). Now, let $\mathcal{P}$ be any partition of $[a, b]$ with $\Delta x_i <$ this δ for all i. By the EVT, on the interval $[x_{i-1}, x_i]$ from the partition $\mathcal{P}$, f attains a maximum $M_i = f(c_i)$ and a minimum $m_i = f(d_i)$ at some $c_i, d_i \in [x_{i-1}, x_i]$. But then $M_i - m_i < \varepsilon/(b - a)$ since $|c_i - d_i| < \delta$. Hence

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \sum_{i=1}^{n} \frac{\varepsilon}{(b - a)} \Delta x_i = \frac{\varepsilon}{b - a} \sum_{i=1}^{n} \Delta x_i = \varepsilon.$$

(This follows since $\sum_i \Delta x_i = b - a$.) Therefore f is integrable on $[a, b]$.

IV. True-False. Say whether each of the following statements is true or false. For true statements, give short proofs; for false ones give reasons or counterexamples. Do any *three* parts. If you submit solutions for all four, then I will consider the other one for Extra Credit.

A) (10) Let $f(x) = e^{2x} - e^x$. There exists some $c \in (0, \ln(2))$ such that $f(c) = 1$.

Solution: The statement is TRUE. We apply the IVT. First, f is continuous everywhere since the exponentials e^x and e^{2x} are differentiable everywhere. On the interval $[0, \ln(2)]$, we have $f(0) = 1 - 1 = 0$ and $f(\ln(2)) = 4 - 2 = 2$. Since 1 is in the range between the endpoint values, the ("weak form" of) the IVT implies that there exists $c \in (0, \ln(2))$ such that $f(c) = 1$.

B) (10) The function $f(x) = \arctan(x)$ is uniformly continuous on the interval $(-1, 1)$.

Solution: This is TRUE. Method 1: $f(x) = \arctan(x)$ is continuous at all $x \in \mathbf{R}$, hence on the closed interval $[-1, 1]$. Our general theorem implies that $f(x)$ is uniformly continuous on $[-1, 1]$, hence also on the subset $(-1, 1)$.

Method 2: We can also apply the MVT to f on the interval $[x, x']$ where $-1 < x < x' < 1$ are arbitary. Then there exists a $c \in (x, x')$ such that $f(x) - f(x') = f'(c)(x - x')$. But $|f'(x)| = \frac{1}{1+x^2}$ is bounded above by 1 on $\mathbf{R}$, hence on this interval. Therefore $|f(x) - f(x')| \leq |x - x'|$, so f is Lipschitz continuous (with Lipschitz constant $k = 1$), hence uniformly continuous.

C) (10) There are continuous functions $f(x)$ on $[a, b]$ for which there exist no differentiable function $F(x)$ on $[a, b]$ with $F'(x) = f(x)$.

Solution: This is FALSE. The Fundamental Theorem of Calculus (part 1) implies that $F(x) = \int_a^x f(t)\, dt$ is always an antiderivative of $f(x)$ on $[a, b]$.

D) (10) Let f be differentiable on an open interval I with $[a, b] \subset I$. If $f'(a) > 0$ and $f'(b) < 0$, then there must exist some $c \in (a, b)$ where $f'(c) = 0$.

Solution: This is TRUE. Since f is differentiable everywhere on $[a, b]$, it is also continuous on that interval. By the Extreme Value Theorem, f reaches a maximum value $M = f(c)$ for c somewhere in that interval. On the other hand, the inequalities $f'(a) > 0$ and $f'(b) > 0$ imply that f must take values larger than both $f(a)$ and $f(b)$ in the interval. For instance, to get

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} > 0,$$

it must be true that $f(x) > f(a)$ for x in some interval $(a, a + \delta)$ for $\delta > 0$. Similarly $f'(b) < 0$ implies $f(x) > f(b)$ for x in some $(b - \delta, b)$. Hence the location where the maximum is attained is $c \in (a, b)$. It follows that $f(c)$ must be a local maximum (not an endpoint maximum) and hence $f'(c) = 0$ since we are assuming f is always differentiable on $I \supset [a, b]$.