Mathematics 242 – Principles of Analysis
Solutions for Exam 2 – March 28, 2014

I.

A) (20) State and prove the Monotone Convergence Theorem for sequences. (You may give the proof in the case that the sequence is monotone increasing.)

Solution: The statement is that any monotone bounded sequence of real numbers converges. If the sequence is monotone increasing, let $a = \text{lub}\{x_n \mid n \in \mathbf{N}\}$. Then for all $\varepsilon > 0$, $a - \varepsilon$ is not an upper bound for $\{x_n \mid n \in \mathbf{N}\}$, so there exist n_0 such that $a - \varepsilon < x_{n_0} \le a$. But then since $\{x_n\}$ is monotone increasing, we have $a - \varepsilon < x_{n_0} \le x_n \le a$ for all $n \ge n_0$. This implies $|x_n - a| < \varepsilon$ for all $n \ge n_0$. Hence $x_n \to a$.

B) (10) Let $\{x_n\}$ be the sequence defined by $x_1 = 1$ and

$$x_{n+1} = \sqrt{3x_n + 1}$$

for all $n \ge 1$. Does this sequence converge? Why? If it does, what is the limit?

Solution: We have $x_1 = 1$ and $x_2 = \sqrt{3 \cdot 1 + 1} = 2$. Assuming $x_{k+1} > x_k$, it follows that

$$x_{k+2} = \sqrt{3x_{k+1} + 1} > \sqrt{3x_k + 1} = x_{k+1}.$$

Therefore, the sequence is monotone (strictly) increasing. Next, we claim that all the terms in the sequence are bounded above by 4. This is true for $n = 1, 2$ by the above. Furthermore, if $x_k < 4$, then $x_{k+1} < \sqrt{3 \cdot 4 + 1} = \sqrt{13} < 4$ so the sequence is bounded above by 4. Therefore, it converges by the Monotone Convergence Theorem. The limit is found by letting $n \to \infty$ in the recurrence. If a denotes the limit then $a = \sqrt{3a + 1}$, so $a^2 - 3a - 1 = 0$. By the quadratic formula, $a = \frac{3+\sqrt{13}}{2}$ (the negative sign gives a number < 0, so that cannot be the limit since $x_n > 0$ for all n).

II.

A) (15) Show using the ε, n_0 definition that

$$\lim_{n\to\infty} \frac{4n+1}{7n+3} = \frac{4}{7}.$$

Solution: Given $\varepsilon > 0$, let $n_0 > \frac{5}{49\varepsilon}$ (such n_0 exist since $\mathbf{N}$ is not bounded in $\mathbf{R}$). This implies that $\frac{5}{49n_0} < \varepsilon$. Then for all $n \ge n_0$,

$$\left|\frac{4n+1}{7n+3} - \frac{4}{7}\right| = \frac{5}{49n+21} < \frac{5}{49n} \le \frac{5}{49n_0} < \varepsilon.$$

This shows the limit is $\frac{4}{7}$.

B) (15) Show using the ε, δ definition that

$$\lim_{x \to 1} \frac{4x+1}{7x+3} = \frac{1}{2}.$$

Solution: Given $\varepsilon > 0$, let $\delta = \min(1, 6\varepsilon)$. Then for all x in the deleted neighborhood of 1 defined by $0 < |x-1| < \delta$ we have $0 < x < 2$, so $\frac{1}{|14x+6|} < \frac{1}{6}$ and $|x-1| < 6\varepsilon$. Therefore

$$\left| \frac{4x+1}{7x+3} - \frac{1}{2} \right| = \frac{|x-1|}{|14x+6|} =< 6\varepsilon \cdot \frac{1}{6} = \varepsilon.$$

This shows the limit is $\frac{1}{2}$.

III. (10) Suppose $\{x_n\}$ is a sequence such that $|x_n - 20| < 50$ for all $n \geq 1$. Show that there exists some number $a \in [-30, 70]$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to a$. State any "big theorems" you are using.

Solution: We have $-30 < x_n < 70$ for all n. This shows the sequence is bounded. Hence the Bolzano-Weierstrass theorem implies it has a convergent subsequence $x_{n_k} \to a$. By the Order Limit theorem, it follows that $-30 \leq a \leq 70$.

IV. Give an example, or give a reason why there can be no such examples:

A) (10) A function f that is continuous at $x = 0$, but not continuous at any other x.

Solution: The function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ -x & \text{if } x \notin \mathbf{Q} \end{cases}$$

is an example. $\lim_{x \to 0} f(x) = 0 = f(0)$, so f is continuous at $x = 0$. However, f is not continuous at any other c, since all neighborhoods of any $c \neq 0$ contain x where $f(x)$ is close to c and other x where $f(x)$ is close to $-c$. Therefore $\lim_{x \to c} f(x)$ does not exist if $c \neq 0$.

B) (10) A sequence x_n such that $x_n \to 3$, but for all $n_0 \in \mathbf{N}$ there exist $n \geq n_0$ with $x_n < 0$.

Solution: There cannot be such a sequence. Let $\varepsilon = 1$, for instance. Then if $x_n \to 3$, there must equal an $n_0 \in \mathbf{N}$ such that $|x_n - 3| < 1$ for all $n \geq n_0$. But $|x_n - 3| < 1$ implies $2 < x_n < 4$ so all the terms for $n \geq n_0$ must be strictly positive.

C) (10) A function f and a c in the domain of f such that $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$, but f is not continuous at c.
Solution: Let

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then for $c = 0$, we have $\lim_{x\to 0^+} f(x) = 1 = \lim_{x\to 0^-} f(x)$. But $f(0) = 0 \neq 1$. Therefore f is not continuous at 0. (This is a *removable discontinuity.*)

Extra Credit. (10) Assume that $\{x_n\}$ is a sequence that converges to a. Construct a new sequence $\{y_n\}$ by making y_n the average of the first n terms in the original sequence: $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Show that $\{y_n\}$ also converges to a.

Solution: Intuitively, the idea is that as n gets big, most of the terms in the numerator of y_n will be getting close to a, but a few terms at the start might not be that close. However, the n in the denominator means that those few terms at the start contribute less and less as $n \to \infty$. But we need a more precise way to say this to get an actual proof. Here's an idea. By the triangle inequality, note that

$$\begin{aligned}
|y_n - a| &= \left|\frac{x_1 + x_2 + \cdots + x_n}{n} - a\right| \\
&= \frac{1}{n}|x_1 + x_2 + \cdots + x_n - na| \\
&\leq \frac{1}{n}\left(|x_1 - a| + \cdots + |x_n - a|\right).
\end{aligned}$$

Since $\{x_n\}$ converges to a, recall that we know the sequence is bounded. This implies that there exists M such that $|x_n - a| \leq M$ for all n as well. Now, given $\varepsilon > 0$, there exists N such that $|x_n - a| < \varepsilon/2$ for all $n \geq N$. So if $n \geq N$, continuing from the last line above,

$$\begin{aligned}
&\leq \frac{1}{n}(|x_1 - a| + \cdots + |x_{N-1} - a|) + \frac{n - N + 1}{n}\frac{\varepsilon}{2} \\
&< \frac{NM}{n} + \frac{\varepsilon}{2}.
\end{aligned}$$

Since we can think of the N that works here as fixed, but n is still allowed to grow, note that we can now find n large enough so that $\frac{NM}{n} < \frac{\varepsilon}{2}$ as well by taking $n > \frac{2NM}{\varepsilon}$. In other words, given $\varepsilon > 0$, we take $n_0 = \max(N, \frac{2NM}{\varepsilon})$. Then $n \geq n_0$ implies $|y_n - a| < \varepsilon$. This shows that $y_n \to a$ as $n \to \infty$.