Mathematics 242 – Principles of Analysis
Solutions for Exam 1 – February 21, 2014

Directions: Do all work in the blue exam booklet (do not place any work that you want to have considered for credit on this sheet). There are 100 regular and 10 extra credit points.

I.

A) (10) *Define:* The real number a is a *least upper bound* of $A \subset \mathbf{R}$, and state the Least Upper Bound Axiom for $\mathbf{R}$.

Solution: a is a least upper bound of A if (1) $a \geq x$ for all $x \in A$, and (2) if $a' \geq x$ for all $x \in A$, then $a' \geq a$. The LUB Axiom states that if A is a nonempty subset of $\mathbf{R}$ that is bounded above, then A has a least upper bound in $\mathbf{R}$.

B) (10) Let

$$A = \bigcap_{n=1}^{\infty} \left(1 + \frac{1}{n}, 4 - \frac{1}{n}\right)$$

Explain why A is bounded and determine the least upper and greatest lower bounds for A.

Solution: When $n = 1$, we have $\left(1 + \frac{1}{n}, 4 - \frac{1}{n}\right) = (2, 3)$. For $n > 1$, $1 + \frac{1}{n} < 2$ and $4 - \frac{1}{n} > 3$. Hence the intersection of all the intervals $\left(1 + \frac{1}{n}, 4 - \frac{1}{n}\right)$ is equal to the interval for $n = 1$: $A = (2, 3)$. Hence $\mathrm{glb}(A) = 2$ and $\mathrm{lub}(A) = 3$.

C) (15) Let A be a bounded subset of $\mathbf{R}$ with $\mathrm{lub}(A) = 2$, and let $B = \{-4x + 3 \mid x \in A\}$. What can be said about $\mathrm{glb}(B)$? Prove your assertion.

Solution: glb)(B) must exist and equal -5. Proof: First, 2 is an upper bound for A, so for all $x \in A$, $x \leq 2$. Hence $-4x \geq -8$, and so $-4x + 3 \geq -5$. Next if b is any lower bound for B, we have $b \leq -4x + 3$ for all $x \in A$, hence $\frac{b-3}{-4} = \frac{3-b}{4} \geq x$ for all x in A. Since $2 = \mathrm{lub}(A)$, this implies $\frac{3-b}{4} \geq 2$, so $3 - b \geq 8$, and hence $b \leq -5$. This shows $\mathrm{glb}(B) = -5$.

II. (15) Let x_n be the sequence defined by the rules $x_1 = 1$ and $x_{n+1} = -\frac{2}{5}x_n + 1$ for all $n \geq 1$. Show by mathematical induction that

$$0 \leq x_n \leq 1 \text{ for all } n \geq 1.$$

Solution: The base case for the induction is $n = 1$ and $x_1 = 1 \geq 0$, so there is nothing else to be proved there. Now for the induction step, assume $0 \leq x_k \leq 1$. Then the $\frac{-2}{5} \leq \frac{-2}{5}x \leq 0$, so $\frac{-2}{5} + 1 \leq \frac{-2}{5}x + 1 \leq 1$, which shows what we want since $\frac{-2}{5} + 1 \geq 0$.

III. Let $x_n = \frac{3n^2}{5n^2+3n+1}$ for all natural numbers $n \geq 1$.

A) (10) Determine $\lim_{n\to\infty} x_n$ intuitively.

Solution: Dividing numerator and denominator by n^2 have

$$x_n = \frac{3}{5 + \frac{3}{n} + \frac{1}{n^2}}$$

so we expect $x_n \to \frac{3}{5}$ as $n \to \infty$.

B) (20) Use the ε, n_0 definition of convergence to prove that $\{x_n\}$ converges to the number you identified in part A.

Solution: Let $\varepsilon > 0$. Since $\mathbf{N}$ is not bounded in the real numbers, there exists n_0 in $\mathbf{N}$ with $n_0 > \frac{2}{5\varepsilon}$, so $\frac{2}{5n_0} < \varepsilon$. Then for any $n \geq n_0$, we have

$$\begin{aligned}
\left| \frac{3n^2}{5n^2 + 3n + 1} - \frac{3}{5} \right| &= \frac{9n + 3}{25n^2 + 15n + 5} \\
&< \frac{10n}{25n^2} \\
&= \frac{2}{5n} \\
&\leq \frac{2}{5n_0} \\
&< \varepsilon.
\end{aligned}$$

This shows $x_n \to \frac{3}{5}$ as $n \to \infty$.

IV. True-False. For each true statement, give a short proof or reason. For each false statement, give an explicit counterexample.

A) (10) If A and B are two nonempty bounded sets of real numbers and $\text{lub}(B) > \text{lub}(A)$, then $y > x$ for all $y \in B$ and all $x \in A$.

Solution: This is FALSE. A counterexample would be something like $A = [0, 1]$ and $B = [-1, 2]$ Then $\text{lub}(A) = 1$ and $\text{lub}(B) = 2$, but B also contains elements smaller than elements in A: $-1 \in B$ but $-1 < x$ for some $x \in A$ (in fact all x in A, but that would not be necessary to get a counterexample).

B) Let

$$A = \{x \in \mathbf{R} \mid 0 < x < 1, \text{ and } x = r\sqrt{2} \text{ for some } r \in \mathbf{Q}\}.$$

Then $\text{lub}(A) = 1$.

Solution: This is TRUE. First, $x < 1$ for all $x \in A$ by definition. Now, let $0 < a < 1$ be any real number. We claim that there are elements of A between a and 1, so that

a cannot be an upper bound for A. To see this, note that by a theorem we proved in class, there must be rational numbers r in the interval $\left(\frac{a}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. So for any such r, $a < r\sqrt{2} < 1$. Hence $r\sqrt{2} \in A$. This shows that a cannot be an upper bound for A. Hence $\mathrm{lub}(A) = 1$.

Extra Credit (10) Is it possible to produce a sequence x_n whose terms include all the positive and negative integers? If so, give an indication how to construct such a sequence. If not, give a reason why there cannot exist such a sequence.

Solution: There is such a sequence, and an example can be constructed like this: Let

$$x_n = \begin{cases} \frac{1-n}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Listing out the first few terms we see

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

so it is clear that every integer will be in the set of terms of the sequence. The positive integers come from n even and the non-positive integers come from n odd.