Mathematics 242 – Principles of Analysis
Exam 1 – February 21, 2014

Directions: Do all work in the blue exam booklet (do not place any work that you want to have considered for credit on this sheet). There are 100 regular and 10 extra credit points.

I.

A) (10) *Define:* The real number a is a *least upper bound* of $A \subset \mathbf{R}$, and state the Least Upper Bound Axiom for $\mathbf{R}$.

B) (10) Let

$$A = \bigcap_{n=1}^{\infty} \left(1 + \frac{1}{n}, 4 - \frac{1}{n}\right)$$

Explain why A is bounded and determine the least upper and greatest lower bounds for A.

C) (15) Let A be a bounded subset of $\mathbf{R}$ with $\text{lub}(A) = 2$, and let $B = \{-4x+3 \mid x \in A\}$. What can be said about $\text{glb}(B)$? Prove your assertion.

II. (15) Let x_n be the sequence defined by the rules $x_1 = 1$ and $x_{n+1} = -\frac{2}{5}x_n + 1$ for all $n \geq 1$. Show by mathematical induction that

$$0 \leq x_n \leq 1 \text{ for all } n \geq 1.$$

III. Let $x_n = \frac{3n^2}{5n^2+3n+1}$ for all natural numbers $n \geq 1$.

A) (10) Determine $\lim_{n\to\infty} x_n$ intuitively.

B) (20) Use the ε, n_0 definition of convergence to prove that $\{x_n\}$ converges to the number you identified in part A.

IV. True-False. For each true statement, give a short proof or reason. For each false statement, give an explicit counterexample.

A) (10) If A and B are two nonempty bounded sets of real numbers and $\text{lub}(B) > \text{lub}(A)$, then $y > x$ for all $y \in B$ and all $x \in A$.

B) (10) Let

$$A = \{x \in \mathbf{R} \mid 0 < x < 1, \text{ and } x = r\sqrt{2} \text{ for some } r \in \mathbf{Q}\}.$$

Then $\text{lub}(A) = 1$.

Extra Credit (10) Is it possible to produce a sequence x_n whose terms include all the positive and negative integers? If so, give an indication how to construct such a sequence. If not, give a reason why there cannot exist such a sequence.