Mathematics 242 – Principles of Analysis
Final Examination
May 9, 2014

Directions

Do all work in the blue exam booklet. There are 200 possible regular points and 10 possible Extra Credit points.

I. Let $A = \{\tan(x) : x \in [0, \pi/4]\}$ and let $B = \{x : 1 < |x| < 3\}$.

A) (10) What is the set $A \cup B$?
B) (10) What is the least upper bound of the set $C = \{|x-2| : x \in A\}$?

II.

A) (10) State the ε, n_0 definition for convergence of a sequence.
B) (20) Identify $L = \lim_{n\to\infty} x_n$ for the sequence

$$x_n = 1 + \frac{(-1)^n}{\sqrt{n}}$$

and prove using the definition that $\lim_{n\to\infty} x_n = L$.

III. (10) Let $n \in \mathbf{N}$ be an integer, and let $x_n = \sin(n)$ and $y_n = \cos(n)$. In this problem you may use without proof any facts about sequences, subsequences, etc. that we proved in class. Does there exist a strictly increasing index sequence n_k of positive integers such that *both* subsequences x_{n_k} and y_{n_k} converge? Prove your assertion.

IV.
A) (10) Give the ε, δ definition for the statement $\lim_{x\to c} f(x) = L$ where $c \in \mathbf{R}$. How does this change if $c = +\infty$?
B) (20) Identify the limit $L = \lim_{x\to 1} x^2 - 4x + 2$ and prove using the definition that the value of L is correct.
C) (10) Identify the limit

$$M = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x^2}}$$

and prove using the definition that M is correct.

V.
A) (20) State and prove the Intermediate Value Theorem.
B) (10) For the rest of this question, let $f(x) = \frac{32x}{x^4+48}$. Show that for each k with $0 < k < 1$, the equation $f(x) = k$ has *at least two* solutions $x \in \mathbf{R}$, with $x > 0$.
C) (10) Show that there are *exactly two* solutions of the equation $f(x) = k$ from part B for each k with $0 < k < 1$.

VI. (20) In this question you may use without proof the summation formulas:

$$\sum_{i=1}^{n} 1 = n \quad \sum_{i=1}^{n} i = \frac{n^2+n}{2} \quad \sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}.$$

Show that

$$f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ -x & \text{if } x \geq 1 \end{cases}$$

is integrable on $[a,b] = [0,2]$ by considering upper and lower sums for f and determine the value of $\int_0^2 f$. (Caution: f is *not monotone* and not continuous on this interval, so you will need to be careful!)

VII. True - False. Do any 4 of the following parts. For each true statement, give a short proof or reason. For each false statement give a reason or a counterexample. (If you submit solutions for more than 4, they will be considered for Extra Credit.)

A) (10) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with real terms. If the partial sums s_N are bounded, $|S_N| \leq B$ for all N, then $\sum_{n=1}^{\infty} a_n$ converges.

B) (10) Let

$$f(x) = \begin{cases} \cos(2x) & \text{if } x < 0 \\ ax^2+bx+c & \text{if } x \geq 0 \end{cases}$$

There is exactly one set of constants a, b, c for which $f'(0)$ and $f''(0)$ both exist.

C) (10) The power series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n} \cdot 2^n}$$

converges absolutely for all $x \in [-2, 2]$.

D) (10) If f is differentiable on an interval $I = (a,b)$ and $f'(x) \neq 0$ for all $x \in I$, then for each k the equation $f(x) = k$ has *at most one* solution with $x \in I$.

E) (10) (Note: This question is suggested by a problem I assigned in my MATH 136 class.) Let n, k be positive integers. The *centroid* of the region R bounded by $y = x^n$ and $y = x^{n+k}$, and $0 \leq x \leq 1$ is the point in $\mathbf{R}^2$ with coordinates

$$(\overline{x}, \overline{y}) = \left(\frac{(n+k+1)(n+1)}{(n+k+2)(n+2)}, \frac{(n+k+1)(n+1)}{(2n+2k+1)(2n+1)} \right).$$

The centroid satisfies $\overline{x}^{n+k} \leq \overline{y} \leq \overline{x}^n$ for each k and n.

Have a safe, enjoyable, and productive summer!