Mathematics 242 – Principles of Analysis
Solutions – Final Examination
May 9, 2014

I. We have

$$A = \{\tan(x) : x \in [0, \pi/4]\} = [0, 1]$$

and

$$B = \{x : 1 < |x| < 3\} = (-3, -1) \cup (1, 3)$$

so:

A) $A \cup B = [0, 1] \cup ((-3, -1) \cup (1, 3)) = (-3, -1) \cup [0, 3)$.

B) For each $x \in A$, $|x - 2|$ represents the distance along the number line from x to 2. So

$$C = \{|x - 2| : x \in A\} = [1, 2]$$

The least upper bound of C is 2.

II. A) We say $\lim_{n\to\infty} x_n = L$ if for all $\varepsilon > 0$, there exist n_0 such that $|x_n - L| < \varepsilon$ for all $n \geq n_0$.

B) We have

$$\lim_{n\to\infty} 1 + \frac{(-1)^n}{\sqrt{n}} = 1.$$

To prove this, note that

$$\left|1 + \frac{(-1)^n}{\sqrt{n}} - 1\right| = \frac{1}{\sqrt{n}}$$

For any given $\varepsilon > 0$, since $\mathbf{N}$ is not bounded in $\mathbf{R}$, there exist

$$n_0 > \frac{1}{\varepsilon^2} \Leftrightarrow \frac{1}{\sqrt{n_0}} < \varepsilon.$$

Hence if $n \geq n_0$, then we have

$$\left|1 + \frac{(-1)^n}{\sqrt{n}} - 1\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n_0}} < \varepsilon.$$

This shows the sequence converges to 1.

III. There is such an index sequence n_k. The sequence $x_n = \sin(n)$ is bounded since $|\sin(n)| \leq 1$ for all n. Hence the Bolzano-Weierstrass Theorem says that there exists a convergent subsequence $x_{n_\ell} = \sin(n_\ell)$ for some strictly increasing index sequence of integers n_ℓ. Consider the sequence $\cos(n_\ell)$, *using these same indices* n_ℓ. This is also a bounded sequence since $|\cos(n_\ell)| \leq 1$ for all ℓ. Hence the Bolzano-Weierstrass Theorem implies that there is a convergent subsequence of this sequence, say $y_{n_{\ell_k}} = \cos(n_{\ell_k})$. Note that this sequence comes from a subsequence of the index sequence n_ℓ. Hence $x_{n_{\ell_k}}$ is a subsequence of a convergent sequence. Since any subsequence of a convergent sequence is convergent too, the $x_{n_{\ell_k}}$ is also convergent.

IV. A) Let f be defined on a deleted interval D about c (that is, at all points on some interval containing c, except possibly at c itself). Then we say $\lim_{x\to c} f(x) = L$ if for all $\varepsilon > 0$, there exist $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in D$ with $0 < |x - c| < \delta$. If we are considering the limit as $x \to +\infty$, then we should assume $f(x)$ is defined for all x sufficiently large and the definition of $\lim_{x\to+\infty} f(x) = L$ becomes: For all $\varepsilon > 0$, there exist $B > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > B$.

B) The limit here is -1. Proof: Let $\varepsilon > 0$ and let $\delta = \min(1, \varepsilon/3)$. Then for all x with $0 < |x - 0| < \delta$, we have

$$|x^2 - 4x + 2 - (-1)| = |x^2 - 4x + 3| = |x - 1||x - 3|.$$

Since $|x - 1| < 1$, we have $0 < x < 2$. Hence $|x - 3| < 3$. But then:

$$|x^2 - 4x + 2 - (-1)| = |x - 1||x - 3| < \frac{\varepsilon}{3} \cdot 3 = \varepsilon.$$

This shows the limit equals $L = -1$.

C) As $x \to \infty$, $1/x \to 0$. Hence the limit should be 1. Let $\varepsilon > 0$, and let $B > \frac{1}{\sqrt{\varepsilon}}$. Then $x > B$ implies $x > \frac{1}{\sqrt{\varepsilon}}$, so $\frac{1}{x^2} < \varepsilon$. Hence if $x > B$, then

$$\left|\frac{1}{1 + \frac{1}{x^2}} - 1\right| = \left|\frac{\frac{-1}{x^2}}{1 + \frac{1}{x^2}}\right| = \frac{1}{x^2 + 1} < \frac{1}{x^2} < \varepsilon.$$

This shows $\lim_{x\to+\infty} \frac{1}{1+\frac{1}{x^2}} = 1$.

V. A) Statement of IVT: If f is continuous on $[a, b]$ and y_0 is any number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ with $f(c) = y_0$. See the class notes or the text for this proof.

B) The denominator $x^4 + 48$ is nonzero for all $x \in \mathbf{R}$. Hence $f(x) = \frac{32x}{x^4+48}$ is continuous at all x. We see $f(0) = 0$ and $f(2) = \frac{64}{64} = 1$. By the IVT on the interval $[0, 2]$, for each k with $0 < k < 1$, there is at least one $x \in (0, 2)$ such that $f(x) = k$. To find a second x satisfying this condition, note that $\lim_{x\to+\infty} f(x) = 0$. Hence given any $0 < k < 1$, we will have $f(b) < k$ for some $b > 2$ as well. By the IVT again, in the interval $[2, b]$, there is also at least one additional solution of $f(x) = k$ for $x \in (2, +\infty)$.

C) If for some k, $f(x_1) = f(x_2) = k$ for some $x_1 \neq x_2$, then $f'(x) = 0$ for some x between x_1 and x_2 by the special case of the MVT known as Rolle's Theorem. However, our function f has derivative

$$f'(x) = \frac{1536 - 96x^2}{(x^4 + 48)^2}$$

(quotient rule for derivatives!). This is zero for $x > 0$ only at $x = 2$. Hence by Rolle's Theorem, on the intervals $(0, 2)$ and $(2, +\infty)$, there cannot be more than one solution of $f(x) = k$ in each interval. This means that there are exactly two of them all together.

VI. We want to show that given any $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[0, 2]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$. Since the function changes from being decreasing to increasing (and is discontinuous) at $x = 1$, let's chose a regular partition of $[0, 1]$ with an *even* number $n = 2q$ of subintervals, so that $x_q = 1$ is always one of the endpoints. We have $x_i = 2i/(2q) = i/q$ for $i = 0, \ldots, 2q$ for each $q \in \mathbf{N}$. On the first half of the interval, f is increasing. So for $i = 1, \ldots, q$, we will have $m_i = f((i-1)/(2q)) = \frac{(i-1)}{2q} + 1$ and for $i = 1, \ldots, q-1$, $M_i = f(i/(2q)) = \frac{i}{2q} + 1$, while $M_q = 2$. But then for $i = q+1, \ldots, 2q$, f is decreasing so $m_i = f(i/(2q)) = \frac{-i}{2q}$, $M_i = f((i-1)/(2q)) = \frac{-(i-1)}{2q}$. At this point, it is possible either to add up the upper and lower sums and subtract, or we can also be more clever. Following the proof that f monotone implies f integrable on $[0, 1]$ and $[1, 2]$, we can see that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = (2-1)\frac{1}{2q} + ((-1) - (-2))\frac{1}{2q} = \frac{1}{q}$$

We can get this $< \varepsilon$ for any $q > 1/\varepsilon$. The value of the integral is computed by taking the limit of the upper sum:

$$\begin{aligned}
\lim_{q\to\infty}\left[\sum_{i=1}^{q}\left(\frac{i}{2q}+1\right) + \sum_{i=q+1}^{2q}\frac{-(i-1)}{2q}\right]\frac{1}{2q} &= \lim_{q\to\infty}\left[\frac{1}{4q^2}\sum_{i=1}^{q} i + \frac{1}{2q}\sum_{i=1}^{q} 1 + \sum_{i=1}^{q}\left(-1-\frac{i}{2q}\right)\frac{1}{2q}\right] \\
&= \lim_{q\to\infty}\left[\frac{1}{4q^2}\frac{q(q+1)}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{4q^2}\frac{q(q+1)}{2}\right] \\
&= 0
\end{aligned}$$

(This can also be checked by the Fundamental Theorem:

$$\int_0^2 f = \int_0^1 x + 1\,dx + \int_1^2 -x\,dx = \frac{x^2}{2} + x\Big|_0^1 - \frac{x^2}{2}\Big|_1^2 = \frac{3}{2} - 2 + \frac{1}{2} = 0.)$$

VII. A) FALSE. Note that the problem *does not say to assume the terms are all positive.* For instance, the series $\sum_{n=1}^{\infty}(-1)^n$ has partial sums

$$S_N = \begin{cases} -1 & \text{if } N \text{ odd} \\ 0 & \text{if } N \text{ even.} \end{cases}$$

So the set of partial sums is bounded with $|S_N| \le 1$ all N. However the sequence of partial sums does not converge, so the series does not converge.

B) TRUE. If $f'(0)$ exists, then f must also be continuous at $x = 0$. This implies

$$\lim_{x\to 0^-}\cos(2x) = 1 = \lim_{x\to 0^+} ax^2 + bx + c.$$

Hence $c = 1$. Then to get $f'(0)$ to exist, we must have

$$\lim_{x \to 0^-} \frac{\cos(2x) - 1}{x} = 0 = \lim_{x \to 0^+} \frac{ax^2 + bx + 1 - 1}{x}$$

This implies $b = 0$. Finally, and similarly, to get $f''(0)$ to exist we must have $a = -2$. Here's why: We have $f'(x) = -2\sin(2x)$ for all $x < 0$ and $f'(x) = 2ax$ for $x > 0$.

$$\lim_{x \to 0^-} \frac{-2\sin(2x) - 0}{x - 0} = -4 = 2a = \lim_{x \to 0^+} \frac{2ax - 0}{x}$$

C) FALSE. The Ratio Test gives that the power series converges absolutely on the open interval $(-2, 2)$. But in fact, note that with $x = 2$ we get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. This series diverges by the integral test:

$$\lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{x}}\, dx = \lim_{b \to \infty} 2\sqrt{b} - 2$$

is not finite. (Or: It's a p-series with $p = 1/2 < 1$, so it must diverge.) Hence the power series does not converge absolutely on the closed interval $[-2, 2]$.

D) TRUE. The idea is the same as in V C above. If there were distinct $x_1 < x_2$ with $f(x_1) = f(x_2)$, then Rolle's Theorem would imply that $f'(c) = 0$ for some $c \in (x_1, x_2)$.

E) FALSE. This can be seen by experimenting a bit with the formulas. For instance with $n = 3$, $k = 1$, we get $\overline{x} = \frac{5 \cdot 4}{6 \cdot 5} = \frac{2}{3}$, and $\overline{y} = \frac{5 \cdot 4}{9 \cdot 7} = \frac{20}{63}$. However $(\overline{x})^3 = \frac{8}{27}$, which is clearly less than $\overline{y}$. Hence $\overline{y} > (\overline{x})^3$.

In fact, we claim that *for each fixed positive integer* k,

$$\lim_{n \to \infty} \overline{y} - (\overline{x})^n = \frac{1}{4} - \frac{1}{e^2} > 0.$$

It follows that there exists an n_0 (depending on k) such that $\overline{y} > \overline{x}^n$ for all $n \geq n_0$. (This says that the centroid of the region R lies above the upper boundary of R when n is sufficiently large(!)) Here's how to see this: First

$$\overline{y} = \frac{(n + k + 1)(n + 1)}{(2n + 2k + 1)(2n + 1)} = \frac{n^2 + (k + 2)n + 1}{4n^2 + (4k + 4)n + 1}$$

and it follows easily that $\lim_{n \to \infty} \overline{y} = \frac{1}{4}$. Now consider

$$\lim_{n \to \infty} \overline{x}^n = \lim_{n \to \infty} \left(\frac{(n + k + 1)(n + 1)}{(n + k + 2)(n + 2)} \right)^n$$

This is a 1^∞ indeterminate form. So we proceed as in a problem on Problem Set 9. We take the logarithm and apply L'Hopital's Rule to evaluate this limit:

$$\begin{aligned}
\lim_{n\to\infty} \ln\left(\frac{(n+k+1)(n+1)}{(n+k+2)(n+2)}\right)^n &= \lim_{n\to\infty} n \ln\left(\frac{(n+k+1)(n+1)}{(n+k+2)(n+2)}\right) \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{(n^2+(k+2)n+1)}{(n^2+(k+4)n+4)}\right)}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\frac{n^2+(k+4)n+4}{n^2+(k+2)n+1} \cdot \frac{2n^2+6n+3k+4}{(n^2+(k+4)n+4)^2}}{\frac{-1}{n^2}} \\
&= -2
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \overline{x}^n = \lim_{n\to\infty} \left(\frac{(n+k+1)(n+1)}{(n+k+2)(n+2)}\right)^n = e^{-2}.$$

This concludes the proof.