Mathematics 242 – Principles of Analysis
Solutions for Problem Set 9 – **Due:** Friday, April 26

'A' Section

1. Let $f(x) = x^2 + 3x + 3$ on $[1, 2]$.

(a) Show that f is integrable on $[1, 2]$ directly using the definition (that is, do not use any general theorems giving criteria for integrability). Hints: Use regular partitions of $[1, 2]$, and the summation rules

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: Consider the regular partition $\mathcal{P}_n$ of the interval $[1, 2]$, so $x_i = 1 + \frac{i}{n}$ for $i = 0, 1, \ldots, n$. We have $f'(x) = 2x + 3 > 0$ for all $x \in [1, 2]$. Hence f is increasing on the interval and thus evaluating at the left endpoints,

$$\begin{aligned} L(f, \mathcal{P}_n) &= \sum_{i=0}^{n-1} \left(\left(1 + \frac{i}{n}\right)^2 + 3\left(1 + \frac{i}{n}\right) + 3 \right) \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{i=0}^{n-1} i^2 + \frac{5}{n^2} \sum_{i=0}^{n-1} i + \frac{7}{n} \sum_{i=0}^{n-1} 1 \\ &= \frac{(n-1)n(2n-1)}{6n^3} + \frac{5(n-1)n}{2n^2} + 7. \end{aligned}$$

Similarly, evaluating at the right endpoints,

$$\begin{aligned} U(f, \mathcal{P}_n) &= \sum_{i=1}^{n} \left(\left(1 + \frac{i}{n}\right)^2 + 3\left(1 + \frac{i}{n}\right) + 1 \right) \frac{3}{n} \\ &= \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \frac{5}{n^2} \sum_{i=1}^{n} i + \frac{7}{n} \sum_{i=1}^{n} 1 \\ &= \frac{n(n+1)(2n+1)}{6n^3} + \frac{5n(n+1)}{2n^2} + 7. \end{aligned}$$

Hence

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{6}{n}.$$

We can make this $< \varepsilon$ for any $\varepsilon > 0$ by letting $n > \frac{6}{\varepsilon}$. Hence f is integrable by our definition.

(b) Determine the value of $\int_1^2 x^2 + 3x + 3 \, dx$.

Solution: The value of the integral is

$$\lim_{n\to\infty} U(f,\mathcal{P}_n) = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} + \frac{5n(n+1)}{2n^2} + 7 = \frac{1}{3} + \frac{5}{2} + 7 = \frac{59}{6}.$$

(This can be checked by the FTC:

$$\begin{aligned}\int_1^2 x^2 + 3x + 3\, dx &= \frac{x^3}{3} + \frac{3x^2}{2} + 3x\Big|_1^2 \\ &= \left(\frac{8}{3} - \frac{1}{3}\right) + \left(6 - \frac{3}{2}\right) + (6-3) \\ &= \frac{59}{6}.)\end{aligned}$$

2. Explain why the following inequalities must be true without evaluating the integrals involved:

(a)

$$0 < \int_0^{\pi/2} \frac{\sin(x)+\cos(x)}{x^3+1}\, dx < \frac{\pi}{\sqrt{2}}$$

Solution: By trigonometric identities, we have $\sin(x)+\cos(x) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$. Hence for all $x \in [0, \pi/2]$,

$$0 < \frac{\sin(x)+\cos(x)}{x^3+1} < \sqrt{2}.$$

Therefore, by Theorem 5.2.5 (b),

$$0 = 0 \cdot \frac{\pi}{2} < \int_0^{\pi/2} \frac{\sin(x)+\cos(x)}{x^3+1}\, dx < \sqrt{2} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{2}}.$$

(b)

$$\frac{1}{2} < \int_0^1 \frac{1+x-x^2}{1+\tan\left(\frac{\pi x}{4}\right)}\, dx < \frac{5}{4}$$

Solution: On the interval $[0,1]$, $1+x-x^2$ has a minimum value of 1 (at the endpoints) and a maximum value of $\frac{5}{4}$ at $x = \frac{1}{2}$. The denominator has a minimum value of 1 at $x = 0$ and increases to 2 at $x = 1$. By plotting, or by considering the derivative of the quotient, it can be seen that the integrand is increasing on a short interval starting at 0 but decreasing the rest of the way to 1, and the minimum value is $\frac{1}{2}$, at $x = 1$. Hence for all $x \in (0,1)$,

$$\frac{1}{2} < \frac{1+x-x^2}{1+\tan\left(\frac{\pi x}{4}\right)} < \frac{5}{4}$$

and the desired inequalities on the integral follow again from Theorem 5.2.5 (b).

3. Let $F(x) = \int_0^x f(t)\, dt$ where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ t+1 & \text{if } 1 < t \le 2 \end{cases}$$

(a) Find an explicit formula for $F(x)$ valid for all $0 \le x \le 2$.

Solution: We have

$$F(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ \frac{x^2}{2} + x - \frac{1}{2} & \text{if } 1 < x \le 2 \end{cases}$$

(Note the constant $\frac{-1}{2}$ in the formula on the second half of the interval ensures that F is continuous on $[0, 2]$.)

(b) Is F differentiable at $x = 1$? Why or why not? Does this contradict the first part of the FTC?

Solution: By the criterion from B 4 on Problem Set 7, F is not differentiable at $x = 1$, since

$$\lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$$

but

$$\lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 + 2x - 3}{2(x-1)} = \lim_{x \to 1^+} \frac{1}{2}(x+3) = 2.$$

This does not contradict the first part of the FTC since f is not continuous on $[0, 2]$. It has a jump discontinuity at $x = 1$ since $\lim_{t \to 1^+} f(t) = 2$, but $\lim_{t \to 1^-} f(t) = 1$.

4. Find the following derivatives using the FTC:

(a) $\frac{d}{dx} \int_0^x \frac{\sin(t)}{t}\, dt$

Solution: The first part of the FTC implies the derivative is $\frac{\sin(x)}{x}$.

(b) $\frac{d}{dx} \int_{-x^2}^{x^3} e^{-u^2}\, du$

Solution: By the first part of the FTC and the chain rule, the derivative is

$$3x^2 e^{-x^6} + 2x e^{-x^4}.$$

'B' Section

1. Let f be integrable on the interval $[a, b]$, and assume $\int_a^b f(x)\, dx > 0$. Show that there exist $k > 0$ and an interval $[c, d] \subseteq [a, b]$ such that $f(x) > k > 0$ for all $x \in [c, d]$.

Solution: If the function is integrable then the definition of integrability implies that for every $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon \tag{1}$$

and

$$L(f,\mathcal{P}) \leq \int_a^b f(x)\ dx \leq U(f,\mathcal{P}). \tag{2}$$

Since the integral is strictly positive, this means that if we take $\varepsilon < \int_a^b f(x)\ dx$, it follows from (1) and (2) that

$$L(f,\mathcal{P}) > U(f,\mathcal{P}) - \varepsilon > \int_a^b f(x)\ dx - \int_a^b f(x)\ dx = 0$$

In other words, the lower sum $L(f,\mathcal{P}) > 0$ for some partition. This can only happen if $m_i = \text{glb}\{f(x) \mid x \in [x_{i-1}, x_i]\} > 0$ over some interval $[x_{i-1}, x_i] \subset [a,b]$ from the partition $\mathcal{P}$. Hence $f(x) \geq m_i > m_i/2 > 0$ for all $x \in [c,d] = [x_{i-1}, x_i]$. We can take $k = m/2$ to get the strict inequality desired.

2. Let f be continuous on $[a,b]$. Show that there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a).$$

(Hint: Look at Theorem 5.2.5 (b).)

Solution: Since f is continous on $[a,b]$, by the Extreme Value Theorem, it attains a maximum M and a minimum m on the interval By the theorem indicated,

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a),$$

so

$$m \leq \frac{1}{b-a}\int_a^b f(x)\ dx \leq M.$$

But then by the Intermediate Value Theorem (in the second version we discussed), it follows that

$$\frac{1}{b-a}\int_a^b f(x)\ dx = f(c)$$

for some $c \in [a,b]$. Hence after multiplying through again by $b-a$, we get:

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

for some $c \in [a,b]$. (Comment: this result is sometimes called the Mean Value Theorem for Integrals. It is related to the method of computing the *average value* of a function f on $[a,b]$ that you may have seen as an application of integration in Calculus 2.)

3. Logarithm and Exponential. In this problem, we will construct the natural logarithm and and the exponential function e^x "from scratch," without relying on intuition about exponentials (as you probably did in calculus). We start by considering the function

$$L(x) = \int_1^x \frac{1}{t}\, dt, \tag{1}$$

for $x > 0$. Note that $\frac{1}{t}$ is continuous on $(0, +\infty)$, hence the FTC applies to show that L is a differentiable function. From calculus you probably recognize that $L(x) = \ln(x)$. We want to show directly that this makes sense and use this function to construct the inverse function $E(x) = L^{-1}(x)$ which is callled $E(x) = e^x$. Why would we proceed "backwards" like this? The issue is that, while $a^{m/n} = (a^{1/n})^m$ makes immediate sense for any positive $a \in \mathbf{R}$ and any rational exponent, what does a^x actually mean if $x \notin \mathbf{Q}$? Instead of trying to define that directly, we will take an end run around the question.

(a) We claim that the function L "has the right property to be a logarithm" – namely that $L(x \cdot x') = L(x) + L(x')$ for all $x, x' > 0$. Prove this by showing that for any constant $x' > 0$, the function $L(xx')$ also satisfies $\frac{d}{dx}L(xx') = \frac{1}{x}$ for all $x > 0$. Deduce that $L(xx') = L(x) + c$ for some constant c, then determine c by substituting an appropriate value for x.

Solution: By the first part of the FTC and the chain rule, we have $\frac{d}{dx}L(xx') = \frac{1}{xx'} \cdot x' = \frac{1}{x}$. Since $L(x)$ also satisfies $L'(x) = \frac{1}{x}$, this implies by one of our corollaries of the Mean Value Theorem that $L(xx') - L(x) = c$ is a constant. By definition $L(1) = 0$, so if we substitute $x = 1$, we get $c = L(x')$. Therefore $L(xx') = L(x) + L(x')$.

(b) Show that $L(x)$ is strictly increasing for $x > 0$, hence is a 1-1 function on the domain $(0, +\infty)$. Hence L has an inverse function $E : \mathbf{R} \to \{x \in \mathbf{R} \mid x > 0\}$.

Solution: Most of this follows since $L'(x) = \frac{1}{x} > 0$ on $(0, +\infty)$, by the first part of the FTC. Hence L is strictly increasing by one of the corollaries of the MVT, hence 1-1.
Comment: To give every detail here, we should also show that the range of L is $\mathbf{R}$ to show that the domain of the inverse function is actually equal to $\mathbf{R}$. To see this, we can "borrow" from the fact that the harmonic series is divergent, like this: We want to show first that $\{L(x) \mid x \in (0, \infty)\}$ is not bounded above. Let $x = n$ be an integer $n > 1$. By considering the lower sum of $\int_1^x \frac{dt}{t}$ for the partition $\mathcal{P} = \{1, 2, \ldots, n\}$ for the interval $[1, n]$, we can see that

$$L(n) = \int_1^n \frac{dt}{t} > \frac{1}{2} + \cdots + \frac{1}{n} = L\left(\frac{1}{t}, \mathcal{P}\right).$$

As $n \to \infty$, though, the right side increases without any bound, since the harmonic series diverges. Hence the numbers $L(n)$ have no upper bound either. On the other hand by part (a) we have

$$0 = L(1) = L\left(n \cdot \frac{1}{n}\right) = L(n) + L\left(\frac{1}{n}\right).$$

Hence the values $L\left(\frac{1}{n}\right)$ are not bounded below either (they go to $-\infty$ as $n \to infty$). Since L is continuous, its range must equal $(-\infty, \infty) = \mathbf{R}$ (by an application of the Intermediate Value Theorem – do you see why?) for the

(c) Show that the inverse function E satisfies the equation $E(x+x') = E(x) \cdot E(x')$, hence E looks like an exponential function.

Solution: By the defintion of an inverse function we have

$$y = E(x) \Leftrightarrow x = L(y).$$

Let y, y' be any two positive real numbers and let $x = L(y)$, $x' = L(y')$, so $y = E(x)$ and $y' = E(x')$. By part (a) we know that

$$L(yy') = L(y) + L(y').$$

Hence

$$E(x)E(x') = yy' = E(L(y) + L(y')) = E(x + x').$$

(d) We define $x = e$ as the unique solution of the equation $L(x) = 1$. Show that $x = e^{m/n}$ satisfies $L(x) = m/n$ for all rational numbers $x = m/n$.

Solution: The given information says $L(e) = 1$. First we claim $L(e^m) = m$ for all $m \in \mathbf{N}$. This follows by induction since $L(e^1) = 1$ by definition, and if $L(e^k) = k$, then $L(e^{k+1}) = L(e^k \cdot e) = L(e^k) + L(e) = k + 1$ by part (a) and the induction hypothesis.
Next, since $L(e^m) + L(e^{-m}) = L(e^m \cdot e^{-m}) = L(1) = 0$, we have $L(e^{-m}) = -m$, which shows that $L(e^m) = m$ for all integers m. Next, by part (a) again, if n is a positive integer, $1 = L(e) = L((e^{1/n})^n) = nL(e^{1/n})$. So $L(e^{\frac{1}{n}}) = \frac{1}{n}$ whenever n is a positive integer. Finally, combining all of these steps we can write any rational number m/n as a fraction in which $n > 0$ and then

$$L(e^{m/n}) = L((e^{\frac{1}{n}})^m) = mL(e^{1/n}) = \frac{m}{n}L(e) = \frac{m}{n}.$$

(e) Show that the inverse function $E(x)$ of $L(x)$ is differentiable and satisfies $E'(x) = E(x)$. Hint: Look at Section 4.4. What happens if you differentiate on both sides of the equation $L(E(x)) = x$?

Solution: Theorem 4.4.4 implies that E is differentiable at all real x, since $L'(x) = \frac{1}{x}$ is never equal to zero. Hence if we differentiate the equation $L(E(x)) = x$ on both sides using the chain rule we get $L'(E(x))E'(x) = 1$. But $L'(u) = \frac{1}{u}$, so this says $\frac{1}{E(x)} \cdot E(x) = 1$ for all x, so $E'(x) = E(x)$. This is the familiar differentiation rule for the exponential function: $\frac{d}{dx}e^x = e^x$.

(f) Show that

$$E(x) = e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

for all $x \in \mathbf{R}$. Hint: Take the logarithm and use L'Hopital's Rule.

Solution: We have

$$L\left(\left(1+\frac{x}{n}\right)^n\right) = nL\left(1+\frac{x}{n}\right) = \frac{L\left(1+\frac{x}{n}\right)}{\frac{1}{n}}.$$

This is the value of a ratio $g(u)/h(u)$ at $u = \frac{1}{n}$. The top is $g(u) = L(1 + ux)$ and the bottom is $h(u) = u$. Both of these are "nice functions" differentiable at $u = 0$. Note that $\lim_{u\to 0} g(u) = 0 = \lim_{u\to 0} h(u)$. So we can apply the version of L'Hopital's Rule developed on Problem Set 8. We have $g'(u) = \frac{x}{1+ux}$ and $h'(u) = 1$. The ratio $g'(u)/h'(u)$ does have a limit as $u \to 0$:

$$\lim_{u\to 0} \frac{g'(u)}{h'(u)} = \lim_{u\to 0} \frac{x}{1+ux} = x.$$

Therefore by L'Hopital's Rule we know

$$\lim_{u\to 0} \frac{g(u)}{h(u)} = x$$

as well. Since $u = \frac{1}{n} \to 0$ as $n \to \infty$, this implies

$$\lim_{n\to\infty} \frac{L\left(1+\frac{x}{n}\right)}{\frac{1}{n}} = x.$$

Since L is differentiable function at all $x > 0$, it also continuous everywhere. Hence

$$L\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) = \lim_{n\to\infty} L\left(\left(1+\frac{x}{n}\right)^n\right) = x$$

This implies

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = E(x) = e^x.$$