Mathematics 242 – Principles of Analysis
Solutions for Problem Set 7 – **due:** Friday, April 12

*'A' Section*

1. Let $f(x) = \frac{15x}{x^4+x^2+1}$. Using the Intermediate Value Theorem,

A) Show: For all $k \in [-5, 5]$, there exist $c \in [-1, 1]$ such that $f(c) = k$.
*Solution:* First, $f(x)$ is a rational function and $x^4 + x^2 + 1 > 1$ for all $x \in \mathbf{R}$, so it follows that $f(x)$ is continuous at all $c \in \mathbf{R}$. We have $f(-1) = -5$ and $f(1) = 5$. Therefore, by the IVT, for all $k \in [-5, 5]$, there exist $c \in [-1, 1]$ such that $f(c) = k$.

B) Show: For all $k$ with $0 < k < 5$, there exist some $c \in (1, \infty)$ such that $f(c) = k$.
*Solution:* We see by the "Big Theorem" on function limits that $\lim_{x\to\infty} f(x) = 0$. Hence if we have any $k$ with $0 < k < 5$, then there exists a $B > 1$ such that $f(B) < k < 5 = f(1)$. Applying the IVT on the interval $[1, B]$, we see there is a $c \in (1, B) \subset (1, \infty)$ such that $f(c) = k$.

C) Show that if $0 < m < 15$, then $f(x) = mx$ has two real solutions other than $x = 0$.
*Solution:* Note that

$$f(x) - mx = \frac{15x}{x^4 + x^2 + 1} - mx = \frac{-mx^5 - mx^3 - (m - 15)x}{x^4 + x^2 + 1}.$$

The numerator factors as

$$-mx\left(x^4 + x^2 + \frac{m - 15}{m}\right).$$

*Method 1:* The polynomial $g(x) = x^4+x^2+\frac{m-15}{m}$ is continuous at all real $x$. Moreover, $g(0) = \frac{m-15}{m} < 0$ (since $m < 15$ while $g(b) > 0$ if $b$ is sufficiently large because the $x^4$ term occurs with a positive coefficient. Therefore, by the IVT, there is a $c$ with $g(c) = 0$ on the interval $(0, b)$. Since $g(x)$ is an even function, we also have $g(-c) = 0$.

*Method 2:* Applying the quadratic formula to the quadratic in $x^2$, the second factor is zero when

$$x^2 = \frac{-1 \pm \sqrt{1 - 4\left(\frac{m-15}{m}\right)}}{2} = \frac{-\sqrt{m} \pm \sqrt{60 - 3m}}{2\sqrt{m}}$$

In order for this to be defined, we must have $m > 0$. For $x^2$ to be positive we must take the + sign and we must have $60 - 3m > m$, so $m < 15$. For $0 < m < 15$, there are two solutions found by taking the positive and negative square roots of the above:

$$x = \pm\sqrt{\frac{-\sqrt{m} + \sqrt{60 - 3m}}{2\sqrt{m}}}.$$

Neither of these is equal to zero.

2. Show that there are at least three real solutions of the equation $\sin(x) + 2\cos(x) = x/2$. Hint: Look at the values of $g(x) = \sin(x) + 2\cos(x) - x/2$ at "nice" multiples of $\frac{\pi}{2}$.

*Solution:* From calculus we know that $\sin(x)$ and $\cos(x)$ are differentiable for all $x$ and the same is true for $x/2$. Therefore, the function $g(x)$ is differentiable everywhere and consequently continuous everywhere. We have

$$\begin{aligned} g\left(\frac{-3\pi}{2}\right) &= 1 + \frac{3\pi}{4} > 0 \\ g(-\pi) &= -2 + \frac{\pi}{2} < 0 \\ g\left(\frac{-\pi}{2}\right) &= -1 + \frac{\pi}{4} < 0 \\ g(0) &= 2 > 0 \\ g\left(\frac{\pi}{2}\right) &= \frac{1}{2} > 0 \\ g(\pi) &= -2 - \frac{\pi}{2} < 0 \end{aligned}$$

Therefore, the IVT implies that $g$ has one zero in the interval $(-3\pi/2, -\pi)$, another in the interval $(\frac{-\pi}{2}, 0)$, and a third in the interval $(\pi/2, \pi)$.

3. Using the definition of the derivative, find the value of $f'(c)$, or say why $f$ is not differentiable at $x = c$:

A) $f(x) = x^3 + 2x + 1$ at $c = 2$.
*Solution:* We have

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{(x^3 + 2x + 1) - 13}{x - 2} \\ &= \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 6)}{x - 2} \\ &= \lim_{x\to 2} x^2 + 2x + 6 \\ &= 14. \end{aligned}$$

B) $f(x) = \sin(|x|)$ at $c = 0$. Hint: Look back at Problem Set 6, B 2.
*Solution:* $f'(0)$ does not exist for this function because

$$\lim_{x\to 0^+} \frac{\sin(|x|)}{x} = +1$$

by the indicated problem on Problem Set 6, while

$$\lim_{x\to 0^-} \frac{\sin(|x|)}{x} = \lim_{x\to 0^-} \frac{-\sin(x)}{x} = -1.$$

Since the one-sided limits are not equal, the derivative at 0 does not exist.

C) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ 2x - 1 & \text{if } x \le 1 \end{cases}$$

at $c = 1$.

*Solution:* We have

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^+} x + 1 = 2.$$

On the other hand,

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{2x - 2}{x - 1} = \lim_{x \to 1^-} 2 = 2.$$

Since the one-sided limits exist and are equal, $f'(1)$ exists and equals 2.

D) The function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

at $c = 0$.

*Solution:* We have for $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

Given any $\varepsilon > 0$, if we take $\delta = \varepsilon$, then for all $x$ in the deleted neighborhood defined by $0 < |x| < \varepsilon$,

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| < \varepsilon.$$

It follows that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0 = f'(0).$$

(It is not too hard to show that $f'(c)$ exists only for this one $c = 0$. This function is not differentiable anywhere else.)

4. Suppose $f, g$ are differentiable functions with $f(g(x)) = \frac{x}{x^2+1}$ and that

$$\begin{array}{ll} g(1) = 1 & g(2) = 4 \\ g'(1) = 2 & g'(2) = -1 \end{array}$$

Determine the equation of the tangent line to the given graph at the given point.

A) $y = f(x)$ at $(1, f(1))$.

*Solution:* First, $f(1) = f(g(1)) = \frac{1}{2}$. By the Chain Rule $(f \circ g)'(1) = f'(g(1))g'(1) = 2f'(1)$. On the other hand, by the Quotient Rule,

$$(f \circ g)'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

So $(f \circ g)'(1) = 0$ and hence $f'(1) = 0$. The tangent line to $y = f(x)$ at $(1, f(1))$ is the horizontal line $y = \frac{1}{2}$.

B) $y = (f \circ g)(x)$ at $(2, (f \circ g)(2))$.

*Solution:* By the computations in part A, $f(g(2)) = \frac{2}{5}$ and $(f \circ g)'(2) = \frac{-3}{25}$. So the tangent line is

$$y = \frac{2}{5} - \frac{3}{25}(x-2).$$

*'B' Section*

1. Let $f$ be continuous on $[0,1]$ with $f(0) < 0$ and $f(1) > 1$. Suppose that $g$ is another continuous function on $[0,1]$ such that $g(0) \geq 0$ and $g(1) \leq 1$. Show that there exists some $c \in (0,1)$ such that $f(x) = g(x)$.

*Solution:* Let $h(x) = f(x) - g(x)$. Since $f, g$ are continuous on $[0,1]$, the same is true for $h$. By the given information, $h(0) = f(0) - g(0) < 0$ and $h(1) = f(1) - g(1) > 0$. Therefore, the IVT implies that $h(c) = 0$ for some $c \in (0,1)$. But then $0 = h(c) = f(c) - g(c)$, so $f(c) = g(c)$.

2. Let $f$ be continuous on $[a,b]$ with $f(a) < k < f(b)$. Here is a variation on our proof of the Intermediate Value Theorem.

A) Let

$$T = \{x \in [a,b] \mid f(x) > k\}.$$

Show that $T$ has a greatest lower bound and that $f(\text{glb}(T)) = k$.

*Solution:* $T$ is contained in the interval $[a,b]$, so it is a bounded subset of $\mathbf{R}$. Then $c = \text{glb}(T)$ exists by the LUB Axiom. Note that $a < c$ since $f(a) < k$. Hence the interval $[a,c)$ is contained in the complement of $T$. If we let $\{x_n\}$ be any sequence contained in $[a,c)$ converging to $c$, then since $f$ is continuous, $\lim_{n\to\infty} f(x_n) = f(c)$. But $f(x_n) \leq k$ for all $n$, so

$$f(c) = \lim_{n\to\infty} f(x_n) \leq k \tag{1}$$

also (by Corollary 2.2.8 in the text). On the other hand, given any $\varepsilon > 0$, $c + \varepsilon$ is not a lower bound for $T$, so there exists some $x \in T$ such that $c \leq x < c + \varepsilon$. Apply this for $\varepsilon = \frac{1}{n}$ for each natural number. Then we get a sequence $x'_n$ such that $x'_n \in T$ for all $n$ and $c \leq x_n' < c + \frac{1}{n}$. It follows easily that $x_n' \to c$ as $n \to \infty$. Therefore since $f$ is continuous at $c$, $\lim_{n\to\infty} f(x_n') = f(c)$. But $x_n' \in T$ for all $n$, so $f(x_n') > k$. Hence

$$f(c) = \lim_{n\to\infty} f(x_n') \geq k. \tag{2}$$

The two inequalities (1) and (2) show that $f(c) = k$.

B) Will this glb($T$) always be the same as the $c$ we found in our proof of the IVT with $f(c) = k$? If so, prove they are the same; if not, give a counterexample.

*Solution:* In the proof we did in class we considered

$$S = \{x \in [a, b] \mid f(x) \leq k\}$$

and we showed that if $c' = \text{lub}(S)$, then $f(c') = k$. The $c$ found in part A and the $c'$ here do not have to be the same. For instance, let $f(x) = x^3 - 2x + 1$ on $[-2, 2]$. We have $f(-2) = -3$ and $f(2) = 5$. So the IVT will apply for any $k$ with $-3 < k < 5$. Consider $k = 0$. The equation $x^3 - 2x + 1 = 0$ actually has three different roots in the interval $[-2, 2]$: One between $-2$ and $-1$ (call this one $\alpha$), a second between $1/2$ and 1 (call this one $\beta$), and a third at $x = 1$. The set $T$ as in part A is the union $T = (\alpha, \beta) \cup (1, 2)$, so $c = \text{glb}(T) = \alpha$. On the other hand, the set $S$ as in the proof we did in class is $S = [-2, \alpha] \cup [\beta, 1]$, so $c' = \text{lub}(S) = 1$.

3. This property deals with another property of real-valued functions of a real variable sometimes called *Lipschitz continuity.*

A) Let $f$ be a function on an interval $I$ with the property that there exists a strictly positive constant $k$ such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in I$ (this is the definition of Lipschitz continuity). Show that $f$ is uniformly continuous on $I$.

*Solution:* Given $\varepsilon > 0$, let $\delta = \varepsilon/k$. Then for any $x, x' \in I$ such that $|x - x'| < \delta = \varepsilon/k$, it follows that

$$|f(x) - f(x')| \leq k|x - x'| < k \cdot \varepsilon/k = \varepsilon.$$

This shows that the definition of uniform continuity is satisfied for $f$ on $I$.

B) The converse of the statement in part A is not true: Show that $f(x) = x^{1/3}$ is uniformly continuous on $[-1, 1]$, but there is no constant $k$ such that $|f(x) - f(x')| \leq k|x - x'|$ for all $x, x' \in [-1, 1]$. Hint: Think slopes of secant lines to the graph $y = x^{1/3}$.

*Solution:* First, $f(x)$ is continuous on $[-1, 1]$, hence it is uniformly continuous by the result of Theorem 3.6.8 (proved in class before Easter break). Let $x' = 0$ and take arbitrary $x > 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^{1/3}}{x} = \frac{1}{x^{\frac{2}{3}}}.$$

But $\lim_{x\to 0^+} \frac{1}{x^{\frac{2}{3}}} = +\infty$. In other words, the value of the difference quotient will get unboundedly large as $x \to 0^+$. Hence there is no single $k$ such that

$$\left|\frac{f(x)-f(0)}{x-0}\right| \leq k$$

for all $x$ in $[-1,1]$. But that shows that there is no $k$ such that $|f(x)-f(0)| \leq k|x-0|$ for all $x$ in $[-1,1]$.

4. Let $f$ and $g$ be differentiable on $(a,c)$ and let $b \in (a,c)$. Assume $f(b) = g(b)$. Define a new function by

$$p(x) = \begin{cases} f(x) & \text{if } x \in (a,b) \\ g(x) & \text{if } x \in [b,c) \end{cases}$$

Show that $p$ is differentiable on $(a,c)$ if and only if $f'(b) = g'(b)$.

*Solution:* Assume first that $f'(b) = g'(b)$. Then

$$\begin{aligned} \lim_{x\to b^-} \frac{p(x)-p(b)}{x-b} &= \lim_{x\to b^-} \frac{f(x)-f(b)}{x-b} \\ &= f'(b) \end{aligned}$$

By our assumption, this is also

$$\begin{aligned} &= g'(b) \\ &= \lim_{x\to b^+} \frac{g(x)-g(b)}{x-b} \\ &= \lim_{x\to b^+} \frac{p(x)-p(b)}{x-b}. \end{aligned}$$

This shows that $p$ is differentiable at $b$ since the one sided limits of the difference quotients for $p$ exist and are equal. Differentiability of $p$ at all $x \neq b$ in $(a,c)$ follows from the way $p(x)$ is defined. At those $x$, the values of $p$ are either the same as the values of $f$ or the values of $g$ on some interval containing $x$. Hence, for instance, if $a < x_0 < b$, then since $p(x) = f(x)$ for all $x$ in an interval containing $x_0$,

$$p'(x_0) = \lim_{x\to x_0} \frac{p(x)-p(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0),$$

since $f$ is differentiable at $x_0$. Similarly, if $b < x_0 < c$, then $p'(x_0) = g'(x_0)$.

Conversely, suppose that $p$ is differentiable at all $x$ in $(a, c)$. This implies in particular that $p$ is differentiable at $x = b$, so

$$\begin{aligned} f'(b) &= \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} \\ &= \lim_{x \to b^-} \frac{p(x) - p(b)}{x - b} \\ &= \lim_{x \to b^+} \frac{p(x) - p(b)}{x - b} \\ &= \lim_{x \to b^+} \frac{g(x) - g(b)}{x - b} \\ &= g'(b). \end{aligned}$$