Mathematics 242 – Principles of Analysis
Problem Set 6 – **due:** Friday, March 22

*'A' Section*

1. Determine whether each of the following limits exists, then prove that your answers are correct using the $\varepsilon, \delta$ definition.

(a) $\lim_{x \to 3} x^2 - 4x + 1$

*Solution:* The limit is $-2$. Proof: Given $\varepsilon > 0$, let $\delta = \min(1, \varepsilon/3)$. For all $x$ with $0 < |x - 3| < \delta < 1$, we have $2 < x < 4$, so $|x - 1| < 3$ and hence

$$\begin{aligned} |x^2 - 4x + 1 - (-2)| &= |x^2 - 4x + 3| \\ &= |x - 3||x - 1| \\ &< \frac{\varepsilon}{3} \cdot 3 \\ &= \varepsilon. \end{aligned}$$

This shows $\lim_{x \to 3} x^2 - 4x + 1 = -2$.

(b) $\lim_{x \to \frac{1}{2}} x + \frac{1}{x}$

*Solution:* The limit is $\frac{5}{2}$. Proof: Given $\varepsilon > 0$, let $\delta = \min\left(\frac{1}{4}, \frac{\varepsilon}{7}\right)$. For all $x$ with $0 < \left|x - \frac{1}{2}\right| < \delta$, we have

$$\begin{aligned} \left|x + \frac{1}{x} - \frac{5}{2}\right| &= \frac{|2x^2 - 5x + 2|}{|2x|} \\ &= \left|x - \frac{1}{2}\right| \cdot \frac{|x - 2|}{|x|} \end{aligned}$$

Since $\left|x - \frac{1}{2}\right| < \delta < \frac{1}{4}$, we see $\frac{1}{4} < x < \frac{3}{4}$, so $|x - 2| < \frac{7}{4}$ and $\frac{1}{|x|} < 4$. It follows that

$$\begin{aligned} \left|x - \frac{1}{2}\right| \cdot \frac{|x - 2|}{|x|} &< \frac{\varepsilon}{7} \cdot \frac{7}{4} \cdot 4 \\ &= \varepsilon. \end{aligned}$$

Hence $\lim_{x \to \frac{1}{2}} x + \frac{1}{x} = \frac{5}{2}$.

(c) $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

*Solution:* The limit is 3. Proof: For $x \neq 2$, we see

$$\frac{x^3 - 8}{x^2 - 4} = \frac{(x^2 + 2x + 4)(x - 2)}{(x - 2)(x + 2)} = \frac{x^2 + 2x + 4}{x + 2}$$

Given $\epsilon > 0$, let $\delta = \min(1, 3\varepsilon/4)$. Then for all $x$ with $0 < |x-2| < \delta < 1$, it follows that $1 < x < 3$, so $|x+1| < 4$ and $\frac{1}{|x+2|} < \frac{1}{3}$. Then

$$\begin{aligned}\left|\frac{x^2+2x+4}{x+2} - 3\right| &= \frac{|x^2-x-2|}{|x+2|} \\ &= |x-2| \cdot \frac{|x+1|}{|x+2|} \\ &< \frac{3\varepsilon}{4} \cdot \frac{4}{3} \\ &= \varepsilon.\end{aligned}$$

This shows $\lim_{x\to 2} \frac{x^3-8}{x^2-4} = 3$.

(d) Let

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

and consider $\lim_{x\to 0} f(x)$.

*Solution:* The limit is 0. Proof: Given $\varepsilon > 0$, let $\delta = \varepsilon$. Then for all $x$ with $0 < |x| < \delta = \varepsilon$, we have

$$\left|x\sin\left(\frac{1}{x}\right) - 0\right| = |x|\left|\sin\left(\frac{1}{x}\right)\right| \leq |x| < \varepsilon$$

(since $|\sin(u)| \leq 1$ for all real $u$). This shows that the limit is 0 as claimed.

2. Which of the functions in question 1 are *continuous* at the indicated $c$ in the limits there? Explain.

*Solution:* The functions in parts (a) and (b) of problem 1 are continuous at the given $c$ since $\lim_{x\to c} f(x) = f(c)$. The function in part (c) is not continuous at 2 since $f(2)$ is not defined. The function in part (d) is not continuous at $x = 0$ since $\lim_{x\to 0} f(x) \neq f(0)$.

3. True-False. For the true statements, give a short proof. For the false statements give a counterexample.

(a) If $\lim_{x\to 1} f(x) = e - \frac{27}{10}$, then there exists a $\delta > 0$ such that $f(x) > 0$ for all $x$ with $0 < |x-1| < \delta$.

This is TRUE. The reason is that $e - \frac{27}{10} > 0$. So if we take $\varepsilon = \left(e - \frac{27}{10}\right)/2$, then there is a corresponding $\delta > 0$ such that for $x$ with $0 < |x-1| < \delta$,

$$\left|f(x) - \left(e - \frac{27}{10}\right)\right| < \left(e - \frac{27}{10}\right)/2.$$

But this implies $f(x) > \left(e - \frac{27}{10}\right)/2 > 0$.

(b) If $|f(x)| \leq x^3$ for all $x$, then $\lim_{x\to 2} f(x) = 8$.

*Solution:* This is FALSE. As a counterexample, we can take the function $f(x) = 0$ for all $x$.

(c) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by this rule:

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ -2x & \text{if } x \text{ is irrational.} \end{cases}$$

Then $\lim_{x\to 0} f(x)$ exists and equals 0.

*Solution:* This is TRUE. Given any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then for all $x$ with $0 < |x - 0| < \delta = \varepsilon/2$, we have $|2x| = |-2x| = 2|x|$ for rational and irrational $x$, so

$$|f(x)| = 2|x| < 2\frac{\varepsilon}{2} = \varepsilon.$$

This shows the limit is 0 as claimed.

(d) If $f(x) < g(x)$ on a deleted neighborhood of $c$, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then $L < M$.

*Solution:* This is FALSE. Counterexample: Let $f(x) = x^4$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x$ with $0 < |x - 0| < 1$. But $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. (The statement would be true if it said $L \leq M$.)

*'B' Section*

1. Assume that $\lim_{x\to c} f(x) = L$.

(a) Show that there exists a constant $B$ and $\delta > 0$ such that $|f(x)| \leq B$ for all $x$ in the deleted neighborhood $\{x \in \mathbf{R} \mid 0 < |x - c| < \delta\}$.

*Solution:* Since $\lim_{x\to c} f(x) = L$, letting $\varepsilon = 1$, there is a corresponding $\delta > 0$ such that $|f(x) - L| < 1$ for all $x$ in the deleted neighborhood defined by $0 < |x - c| < \delta$. But for those $x$, $L - 1 < f(x) < L + 1$, so $|f(x)| \leq \max(|L + 1|, |L - 1|)$. We can take $B = \max(|L + 1|, |L - 1|)$.

(b) Using part (a), *not* the limit product rule, show that $\lim_{x\to c}(f(x))^n = L^n$ for all integers $n \geq 1$.

*Solution:* Let $B$ and $\delta_0$ be as in part (a). That is assume that $|f(x)| \leq B$ for all $x$ with $0 < |x - c| < \delta_0$. Given $\varepsilon$, since $\lim_{x\to c} f(x) = L$, we have $|f(x) - L| < \varepsilon/M$, where

$$M = B^{n-1} + B^{n-2}|L| + \cdots + B|L|^{n-2} + |L|^{n-1}$$

for all $x$ with $0 < |x - c| < \delta_1$ for some $\delta_1 > 0$. Let $\delta = \min(\delta_0, \delta_1)$. Then for all $x$ with $0 < |x - c| < \delta$, we have (using the triangle inequality on the second factor on the right side):

$$\begin{aligned}|(f(x))^n - L^n| &= |f(x) - L||(f(x))^{n-1} + (f(x))^{n-2}L + \cdots + f(x)L^{n-2} + L^{n-1}| \\ &\leq |f(x) - L|(|f(x)|^{n-1} + |f(x)|^{n-2}|L| + \cdots + |f(x)||L|^{n-2} + |L|^{n-1}) \\ &< \frac{\varepsilon}{M}(B^{n-1} + B^{n-2}|L| + \cdots + B|L|^{n-2} + |L|^{n-1}) \\ &= \frac{\varepsilon}{M} \cdot M = \varepsilon.\end{aligned}$$

This shows $\lim_{x\to c}(f(x))^n = L^n$.

(c) Assume that $f(x) \geq 0$ on some deleted neighborhood of $x = c$. Show that

$$\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}.$$

(*Hint:* It may help to treat the cases $L = 0$ and $L \neq 0$ separately.)

*Solution:* First suppose $L = 0$. Then for all $\varepsilon > 0$, there exist corresponding $\delta > 0$ such that $|f(x)| < \varepsilon^2$ for all $x$ with $0 < |x - c| < \delta$. But then for the same $x$, we have $|\sqrt{f(x)}| < \varepsilon$. So $\lim_{x\to c} \sqrt{f(x)} = 0 = \sqrt{0}$. Now assume $L \neq 0$ (so $L > 0$). Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon\sqrt{L}$ for all $x$ with $0 < |x - c| < \delta$. For these $x$,

$$\begin{aligned}|\sqrt{f(x)} - \sqrt{L}| &= \frac{|f(x) - L|}{f(x) + \sqrt{L}} \\ &\leq \frac{|f(x) - L|}{\sqrt{L}} \\ &< \frac{\varepsilon\sqrt{L}}{\sqrt{L}} \\ &= \varepsilon.\end{aligned}$$

This shows $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

2. In this problem you will show that

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

For $0 < \theta < \frac{\pi}{2}$, the point $P = (\cos(\theta), \sin(\theta)) = (x, y)$ lies on the arc of the unit circle $x^2 + y^2 = 1$ in the first quadrant.

(a) Let $O = (0, 0)$, $Q = (\cos(\theta), 0)$, and $R = (1, 0)$. (Draw a picture!) By considering the areas of the triangle $\Delta OQP$ and the circular sector $ORP$, deduce that if $0 < \theta < \frac{\pi}{2}$, then $\sin(\theta)\cos(\theta) \leq \theta$. (You may use "intuitively reasonable" facts about areas such as the statement that if one plane region $\mathcal{R}$ is completely completely contained in a second region $\mathcal{S}$, then $\text{area}(\mathcal{R}) \leq \text{area}(\mathcal{S})$.)

*Solution:* The area of the triangle $\Delta OQP$ is $\frac{1}{2}\sin(\theta)\cos(\theta)$. The area of the sector is $\frac{1}{2}\theta$, since the area of the circular sector with angle $\Theta$ of a circle of radius $r$ is $\frac{\Theta r^2}{2}$. Since the sector completely contains the triangle, the desired inequality follows.

(b) Now take the tangent line to the circle at $R$ (a vertical line), and let $S = (1, \tan(\theta))$ be the intersection of that line and the radius $OP$ (extended). Considering the areas of the triangle $\Delta ORS$ and the sector $ORP$ as above, explain why $\theta \leq \tan(\theta)$.

*Solution:* The area of the triangle $\Delta ORS$ is $\frac{1}{2}\tan(\theta)$. (By the way, in case you never have seen this before, this is the reason that the tangent function is known by that name!) This time the sector $ORP$ is completely contained in the triangle, so the inequality follows again.

(c) Combine parts (a) and (b) to deduce that if $0 < \theta < \frac{\pi}{2}$, then

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}.$$

*Solution:* Combining parts (a) and (b), we see

$$\sin(\theta)\cos(\theta) \leq \theta \leq \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

Since $\sin(\theta) > 0$ for the $\theta$ in this range, it follows that

$$\cos(\theta) \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

and the desired inequalities follow by taking reciprocals.

(d) Using the one-sided form of Theorem 3.2.9 (The Limit Squeeze Theorem), show that

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1.$$

(You will need to use the fact that $\cos(\theta)$ is continuous at $\theta = 0$.)

*Solution:* Since $\cos(\theta)$ is continuous at $\theta = 0$, we have $\lim_{\theta \to 0^+} \cos(\theta) = 1$ and hence $\lim_{\theta \to 0^+} \frac{1}{\cos(\theta)} = 1$ as well. By the one-sided version of the Limit Squeeze Theorem,

$$\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$$

also.

(e) Now, for $-\frac{\pi}{2} < \theta < 0$, show that $\frac{\sin(\theta)}{\theta} = \frac{\sin(|\theta|)}{|\theta|}$ and use this to see that

$$\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = 1$$

as well.

*Solution:* This follows from the fact that sin is an odd function. If $\theta < 0$, then

$$\frac{\sin(|\theta|)}{|\theta|} = \frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta}.$$

By letting $\varphi = |\theta| > 0$, from part (d) we see that

$$\lim_{\theta \to 0^{-1}} \frac{\sin(\theta)}{\theta} = \lim_{\theta \to 0^-} \frac{\sin(|\theta|)}{|\theta|} = \lim_{\varphi \to 0^+} \frac{\sin(\varphi)}{\varphi} = 1.$$

(f) Finally, explain how parts (d) and (e) combine to show the statement at the start of the problem.

*Solution:* The desired statement follows from parts (d) and (e) and Theorem 3.3.4 (equality of the two one-sided limits implies the two-sided limit exists and equals the common value).