Mathematics 242 – Principles of Analysis
Solutions for Problem Set 5
Due: March 15, 2013

'A' Section

1. For each of the following sequences, determine three different subsequences, each converging to a different limit. For each one, express your three subsequences as x_{n_k} for a suitably chosen (strictly increasing) index sequence n_k, and give an explicit formula for n_k as a function of k:

(a) $x_n = \cos\left(\frac{\pi\sqrt{n}}{2}\right)$

Solution: If $n_k = (4k)^2$, then $x_{n_k} = 1$ for all k, so that subsequence converges to 1. If $n_k = (2(2k+1))^2$, then $x_{n_k} = -1$ for all k, so that subsequence coverges to -1. Finally, if $n_k = (2k-1)^2$, then $x_{n_k} = 0$ for all k so that subsequence converges to 0. There are infinitely many other correct examples too.

(b) $x_n = \frac{n}{3} - \left[\frac{n}{3}\right]$ (as usual, [ ] denotes the greatest integer function)

Solution: Let $n_k = 3k$, then $x_{3k} = k - k = 0$ for all k, so that subsequence converges to 0. Let $n_k = 3k+1$, then $x_{3k+1} = k + 1/3 - k = 1/3$ for all k, so that subsequence converges to $1/3$. Finally, if $n_k = 3k+2$, then the subsequence x_{3k+2} converges to $2/3$.

2. Let $x_n = n^{1/4}$. For each of the following sequences, either express that sequence as a subsequence of the sequence x_{n_k} for some explicit (strictly increasing) index sequence n_k, or say why that is impossible:

(a) $\{2, 3, 4, 5, \ldots\}$

Solution: This is the subsequence x_{n_k} for $n_k = (k+1)^4$, $k \geq 1$.

(b) $\{\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots\}$

Solution: This is the subsequence x_{n_k} for $n_k = (3k)^2$, $k \geq 1$.

(c) $\{1, 2, 4, 8, 16, 32, \ldots\}$

Solution: This is the subsequence x_{n_k} for $n_k = 2^{4k-4}$, $k \geq 1$.

3. Let $x_n = \sin\left(\frac{n\pi}{4}\right)$ and $y_n = \cos\left(\frac{n\pi}{4}\right)$.

(a) Find a (strictly increasing) index sequence n_k such that both x_{n_k} and y_{n_k} converge.

Solution: The sequence $n_k = 8k+1$ is such an index sequence since

$$\sin\left(\frac{(8k+1)\pi}{4}\right) = \sin\left(2k\pi + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and

$$\cos\left(\frac{(8k+1)\pi}{4}\right) = \cos\left(2k\pi + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

for all $k \geq 1$.

(b) Find a second (strictly increasing) index sequence n_k such that both x_{n_k} and y_{n_k} diverge.

Solution: The sequence $n_k = k$ is such a sequence.

(c) Find a third (strictly increasing) index sequence n_k such that one of x_{n_k} and y_{n_k} converges and the other diverges.

Solution: The sequence $n_k = 4k$ is such an example since $\sin(k\pi) = 0$ for all k, but $\cos(k\pi) = (-1)^k$ does not converge.

'B' Section

1. (True or False) – If the statement is true give a proof; if it is false give a counterexample.

(a) If x_n is a sequence of strictly negative numbers converging to 0, then x_n has a strictly increasing subsequence x_{n_k}.

Solution: This is TRUE. Here is one way to see it, constructing a strictly increasing subsequence inductively. We start with $n_1 = 1$, so $x_{n_1} = x_1$. This is the base case. Now assume we have found $x_{n_1} < x_{n_2} < \ldots < x_{n_k}$ for $1 = n_1 < n_2 < \cdots < n_k$. Since $x_n \to 0$, given $\varepsilon = |x_{n_k}|$, there exists N_0 in $\mathbf{N}$ such that $|x_n - 0| < |x_{n_k}|$ for all $n \geq N_0$. Take $n_{k+1} = \max(N_0, n_k + 1)$. Then we have $|x_{n_{k+1}}| < |x_{n_k}|$, so $x_{n_{k+1}} > x_{n_k}$ (since they are both negative) and $n_{k+1} > n_k$. This shows we can continue to construct a strictly increasing subsequence.

(b) If $x_n \to 0$, then x_n contains a strictly increasing subsequence or a strictly decreasing subsequence (or both).

Solution: This is FALSE. Counterexamples are the constant sequence $x_n = 0$ for all $n \geq 1$, or any "eventually constant" sequence with $x_n = 0$ for all $n \geq n_0$ for some $n_0 \in \mathbf{N}$. (Comment: The statement would be true if we assumed $x_n \neq 0$ for all n (or even all $n \geq n_0$ for some $n_0 \in \mathbf{N}$). Any such sequence contains either infinitely many positive terms or infinitely many negative terms, or both. If there are infinitely many negative terms, we get a strictly increasing subsequence of the negative terms by part (a) of this question. If there are infinitely many strictly positive terms, then there is a strictly decreasing subsequence, as can be seen by taking negatives, using part (a), then flipping signs again.)

(c) If x_n is a decreasing sequence with a bounded subsequence x_{n_k}, then x_n converges.

Solution: This is TRUE. Since the whole sequence is decreasing, so is the subsequence x_{n_k}. But that subsequence is bounded (below), say by $a \in \mathbf{R}$. We claim that the whole sequence x_n is also bounded below by a. To see that, let n be any natural number. Since $\{n_k\}$ is a strictly increasing sequence of natural numbers, it follows that it is not bounded above. So there is some k such that $n_k \geq n$. But then since x_n is decreasing and a is a lower bound for the subsequence, $a \leq x_{n_k} \leq x_n$. It follows that $x_n \geq a$ for all n, and hence the whole sequence is bounded below. Then $\{x_n\}$ converges as well by the Monotone Convergence Theorem. (You can also show that the limit of the whole sequence must be the same as the limit of the subsequence, but that was not required.)

2. Consider the sequence $x_n = \cos(n)$ (where we think of n as an angle expressed in radians).

(a) Prove that x_n has a convergent subsequence.

Solution: Since $|\cos(n)| \leq 1$ for all $n \geq 1$, this is a bounded sequence. The statement to be proved is a direct consequence of the Bolzano-Weierstrass Theorem.

(b) In this part of the question we will show that x_n is not convergent, though. Suppose $\lim_{n\to\infty} \cos(n) = a$ for some real number a. Using a trig identity for $\cos(n+1)$ and considering $\lim_{n\to\infty}(\cos(n+1) - \cos(n))$, show that

$$\frac{a(\cos(1) - 1)}{\sin(1)} = \lim_{n\to\infty} \sin(n).$$

But then use the sequence $\lim_{n\to\infty}(\sin(n+1) - \sin(n))$ to deduce that $a = 0$, so $\lim_{n\to\infty} \cos(n) = \lim_{n\to\infty} \sin(n) = 0$. But this is a contradiction. Explain why to conclude the proof.

Solution: The addition formula for cos implies that $\cos(n+1) = \cos(n)\cos(1) - \sin(n)\sin(1)$. If we assume that $\lim_{n\to\infty} \cos(n) = a$, then using parts of the "Big Theorem" and rearranging algebraically, we see

$$\begin{aligned} 0 &= \lim_{n\to\infty} (\cos(n+1) - \cos(n)) \\ &= \lim_{n\to\infty} \cos(n)(\cos(1) - 1) - \lim_{n\to\infty} \sin(n)\sin(1) \\ &= a(\cos(1) - 1) - \sin(1) \lim_{n\to\infty} \sin(n). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \sin(n) = \frac{a(\cos(1) - 1)}{\sin(1)}. \tag{1}$$

as claimed. Now the addition formula for sin shows

$$\sin(n+1) = \sin(n)\cos(1) + \cos(n)\sin(1)$$

Taking the limit as $n \to \infty$ on both sides and substituting from (1), we get:

$$\frac{a(\cos(1) - 1)}{\sin(1)} = \frac{a(\cos(1) - 1)}{\sin(1)} \cos(1) + a\sin(1),$$

so

$$a(\cos(1) - 1) = a(\cos(1) - 1)\cos(1) + a\sin^2(1),$$

and hence (because $\cos^2(1) + \sin^2(1) = 1$),

$$a(\cos(1) - 1) = a(1 - \cos(1)).$$

The only way this can be true is if $a = 0$. But then this implies that

$$\lim_{n\to\infty} \sin(n) = 0 = \lim_{n\to\infty} \cos(n).$$

But that is impossible because it would say $\lim_{n\to\infty} \cos^2(n) + \sin^2(n) = 0$ by parts (a) and (b) of the "Big Theorem." However, we know by the basic identity for the trigonometric functions that

$$\cos^2(n) + \sin^2(n) = 1$$

for all $n \geq 1$. Hence $\lim_{n\to\infty} \cos^2(n) + \sin^2(n) = 1$ as well if the two limits exist. This contradiction shows that $\lim_{n\to\infty} \cos(n)$ cannot exist.

3. A *cluster point* of a sequence x_n is a limit of a convergent subsequence x_{n_k}. (See question 1 on the A section for examples of sequences with several different cluster points.)

(a) Show that there exists a sequence x_n whose set of cluster points is all of $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x > 0\}$. (Hint: Look at the solution for the Extra Credit problem on Exam 1, which shows how to get a sequence containing all the positive rational numbers. Think rational approximations to decimals to get a subsequence converging to any given positive $a \in \mathbf{R}$. However, note that you will need to be careful to produce an actual subsequence x_{n_k} with a strictly increasing index sequence n_k. In fact, this example can be "jazzed up" to get a sequence whose set of cluster points consists is all of $\mathbf{R}$!)

Solution: As shown in the solution to the Extra Credit problem there is a sequence $\{x_n\}$ whose terms contain all the positive fractions p/q with p, q integers ≥ 1. We claim that *every positive real number is cluster point of that sequence.* Note that because of the "zig-zag" way that sequence is constructed, in the sequence $\{x_n\}$, all the fractions p/q with $p + q = \ell$ occur before those with $p + q = \ell + 1$, etc. In other words, the sum of the numerator and the denominator of x_n is an increasing function of n. Now, if p/q is a positive fraction in lowest terms, then we can also consider the fractions $(pk)/(qk)$ for all $k \geq 1$. These form a subsequence of the x_n sequence since the sum $pk + qk = (p + q)k$ is strictly increasing as a function of k. Hence the indices n_k that give $(pk)/(qk) = x_{n_k}$ must also be strictly increasing. That subsequence converges to $a = p/q$ since all the fractions $(pk)/(qk)$ reduce to p/q, so it's actually a constant sequence. Next, any positive irrational number can be expressed with an infinite (nonrepeating) decimal expansion

$$a = d_k d_{k-1} \ldots d_0.f_1 f_2 \cdots$$

(where the d_i and the f_j are the decimal digits). As we saw in class, that expansion is equivalent to a monotone increasing sequence of rational approximations with denominators that are powers of 10:

$$d_k \dots d_0/1, d_k \dots d_0 f_1/10, d_k \dots d_0 f_1 f_2/100, \dots .$$

The notation means something like this: If $a = \sqrt{2}$, for instance, where $\sqrt{2} = 1.4142\cdots$, then our sequence of rational approximations is

$$1/1, 14/10, 141/100, 1414/1000, \dots .$$

When we add one more decimal digit to get the next term in a sequence approximating a given positive real a, the corresponding fraction will definitely occur *later* in the x_n sequence, since the sum of the numerator and the denominator always increases when we include an additional digit. This shows that we have a subsequence converging to a. Hence a is a cluster point of the x_n sequence.

(b) Show that if a_m is a convergent sequence of cluster points of a given sequence x_n, then $a = \lim_{m\to\infty} a_m$ is also a cluster point of the x_n sequence.

Solution: We let a_m be a convergent sequence of cluster points of x_n with $a = \lim_{m\to\infty} a_m$. Since each a_m is a cluster point of the x_n sequence, there is a subsequence of x_n converging to a_m. From the subsequence converging to a_1, select any x_{n_1} with $|x_{n_1} - a_1| < 1$. Then from the subsequence converging to a_2, select any x_{n_2} with $n_2 > n_1$ and $|x_{n_2} - a_2| < \frac{1}{2}$, then from the subsequence converging to a_3, select x_{n_3} with $|x_{n_3} - a_3| < \frac{1}{3}$ and $n_3 > n_2$. By an induction argument, we can always continue this process since for any $\ell \geq 1$, there are infinitely many index values n_ℓ for which $|x_{n_\ell} - a_\ell| < \frac{1}{\ell}$ (all the indices giving terms in the subsequence converging to a_ℓ that are distance $\frac{1}{\ell}$ or less from a_ℓ). In this way we get a subsequence $\{x_{n_\ell}\}$ (indexed by $\ell \in \mathbf{N}$) with n_ℓ strictly increasing and $|x_{n_\ell} - a_\ell| < \frac{1}{\ell}$ for all $\ell \geq 1$. We claim that the subsequence x_{n_ℓ} converges to a. To see this note that given any $\varepsilon > 0$, there exists $\ell_0 \in \mathbf{N}$ such that $\frac{1}{\ell} < \frac{\varepsilon}{2}$ and $|a_\ell - a| < \frac{\varepsilon}{2}$ for all $\ell \geq \ell_0$. Then by the triangle inequality, for all $\ell \geq \ell_0$,

$$|x_{n_\ell} - a| = |x_{n_\ell} - a_\ell + a_\ell - a| \leq |x_{n_\ell} - a_\ell| + |a_\ell - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the subsequence $\{x_{n_\ell}\}$ converges to a, so a is also a cluster point of the $\{x_n\}$ sequence.