MATH 242 – Principles of Analysis
Solutions for Problem Set 2 – due: Feb. 8

'A' Section

1. Let $x \in [1,3]$. Determine the largest and smallest values of $|x-5|$, $|x+5|$, and $1/|x^2-25|$.
   Solution: If $x \in [1,3]$, then $x-5 \in [-4,-2]$, so the largest and smallest values of $|x-5|$ are 4 and 2 respectively. Next, $x+5 \in [6,8]$, so the largest and smallest values of $|x+5|$ are 8 and 6 respectively. Finally, $|x^2-25| = |x-5||x+5|$, so the largest and smallest values of $1/|x^2-25|$ are $1/16$ and $1/24$ respectively. (Think of the graph $y = 1/|x^2-25$ to see this.)
2. Use the binomial theorem (Theorem 1.4.1) for all parts of this problem.
   a. Expand using the binomial theorem and simplify as much as possible:
   $$(a^2-5b^3)^6.$$
   Solution: We have
   $$(a^2-5b^3)^6 = a^{12}-30a^{10}b^3+375a^8b^6-2500a^6b^9+9375a^4b^{12}-18750a^2b^{15}+15625b^{18}.$$
   b. What is the coefficient of x^3 in the expansion of
   $$\left(\frac{x^4+7x}{x^2}\right)^3.$$
   Solution: Dividing inside the power, we are looking at
   $$\left(x^2+7x^{-1}\right)^3$$
   The only x^3 term will come from squaring the x^2 and taking x^{-1} to the first power. So the coefficient is $\binom{3}{1} \cdot 7 = 21$.
   c. What is $\sum_{k=0}^{n} \binom{n}{k}$? Explain.
   Solution: From the binomial theorem this sum is what we obtain from
   $$(1+1)^n = 2^n.$$
   d. What is $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$? Explain.
   Solution: From the binomial theorem this sum is what we obtain from
   $$(1-1)^n = 0^n = 0.$$

3. For each of the following statements, say whether the statement is true or false. If it is false, give a counterexample; if it is true, give a short reason.
   a. A set $A \subset \mathbf{R}$ is bounded if there exists some B such that $x \leq B$ for all $x \in A$.
      Solution: FALSE – the set is bounded above, but not necessarily below. An example would be the open interval $(-\infty, 0)$ with $B = 0$.
   b. If $A, B \subset \mathbf{R}$ are bounded, then $A \cup B$ is also bounded.
      Solution: TRUE – If A is bounded above by M_A and B is bounded above by M_B, then $A \cup B$ is bounded above by $\max(M_A, M_B)$. Similarly, if A is bounded below by m_A and B is bounded below by m_B, then $A \cup B$ is bounded above by $\min(m_A, m_B)$.
   c. If $A, B \subset \mathbf{R}$ are bounded, then $D = \{x - y \mid x \in A, y \in B\}$ is also bounded.
      Solution: TRUE – Suppose A is bounded above by M_A and B is bounded above by M_B. Similarly, suppose A is bounded below by m_A and B is bounded below by m_B. Then for all $x \in A$ and $y \in B$ we have $m_A \leq x \leq M_A$ and $m_B \leq y \leq M_B$. It follows that $m_A - M_B \leq x - y \leq M_A - m_B$. Therefore D is bounded.
   d. If $A, B \subset \mathbf{R}_{>0}$ are bounded, then $Q = \{x/y \mid x \in A, y \in B\}$ is also bounded.
      Solution: FALSE – Let $B = (0, 1)$ and let $A = \{1\}$. The set Q is the set $\{1/y \mid y \in (0, 1)\}$ which is not bounded above.

4.
   a. Let $A = [0, 4] \cap (1, 5)$. What is $a = \text{lub } A$? What is $b = \text{glb } A$? Are $a, b \in A$?
      Solution: We have $A = (1, 4]$. Therefore $a = 4$, which is in A. But $b = 1$ is not in A.
   b. Let $B = \{x \in \mathbf{R} \mid 0 < x^2 - 4x + 1 < 4\}$. What is $a = \text{lub } B$? What is $b = \text{glb } B$? Are $a, b \in B$?
      Solution: By completing the square or using the quadratic formula, we see $B = (2 - \sqrt{7}, 2 - \sqrt{3}) \cup (2 + \sqrt{3}, 2 + \sqrt{7})$. Hence $a = 2 + \sqrt{7}$ and $b = 2 - \sqrt{7}$. Neither is in B.
   c. Let $C = \{x \in \mathbf{Q} \mid x^2 < 5\}$. What is $a = \text{lub } C$? What is $b = \text{glb } C$? Are $a, b \in C$?
      Solution: There are rational numbers arbitrarily close to $\pm\sqrt{5}$, so whose squares are arbitrarily close to 5, but $\pm\sqrt{5}$ are not themselves rational numbers. It follows that $a = \sqrt{5}$ and $b = -\sqrt{5}$, but neither is in C.

'B' Section

1. Let x, y be any real numbers.

a. Show that $|x| - |y| \le |x - y|$ and deduce that $||x| - |y|| \le |x - y|$.

Solution: From the usual triangle inequality,

$$|x| = |(x - y) + y| \le |x - y| + |y|.$$

Subtracting, we obtain $|x| - |y| \le |x - y|$ as desired. Similarly, reversing the roles of x, y, we have $|y| - |x| \le |y - x| = |x - y|$. Since either $|x| \ge |y|$ or $|y| \ge |x|$ is true, we have either $||x| - |y|| = |x| - |y|$ or $||x| - |y|| = |y| - |x|$. Since both of those are $\le |x - y|$, it follows that $||x| - |y|| \le |x - y|$ as desired.

b. Show that if $x, y > 0$, then $x < y$ is equivalent to $x^2 < y^2$.

Solution: Since $x > 0$, $x < y$ implies $x^2 < xy$. Similarly, $x < y$ implies $xy < y^2$. But then transitivity implies $x^2 < y^2$. Conversely, if $x^2 < y^2$, then $0 < y^2 - x^2 = (y - x)(y + x)$. Since $x, y > 0$, the sum $x + y > 0$. This implies that $y - x > 0$ too, or equivalently $x < y$.

c. Show that if $0 < x < y$, then $\sqrt{y} - \sqrt{x} < \sqrt{y - x}$.

Solution: By part b, since $\sqrt{y} - \sqrt{x} > 0$ and $\sqrt{y - x} > 0$, it suffices to show that

$$(\sqrt{y} - \sqrt{x})^2 < (\sqrt{y - x})^2.$$

But the left side here is $y - 2\sqrt{y}\sqrt{x} + x$ and the right side is $y - x$. We have

$$(y - x) - (y - 2\sqrt{y}\sqrt{x} + x) = 2\sqrt{y}\sqrt{x} - 2x = 2\sqrt{x}(\sqrt{y} - \sqrt{x}).$$

This is > 0 because of the assumption $y > x$ and part b. Hence the desired inequality follows.

2. Let a, b be any real numbers. Define $\max(a, b)$ and $\min(a, b)$ to be the larger and smaller of the two numbers, respectively. (That is, $\max(a, b) = a$ if $a \ge b$ and $\max(a, b) = b$ if $b \ge a$. Similarly for the minimum.) Show that

$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}$$

and

$$\min(a, b) = \frac{a + b}{2} - \frac{|a - b|}{2}.$$

Solution: Geometrically, the average $\frac{a+b}{2}$ is the midpoint of the line segment along $\mathbf{R}$ between a and b and $|a - b|$ is the distance between a and b (the length of the line segment). So starting at the midpoint and going $1/2$ the distance to the right gives the maximum of the endpoints, and going $1/2$ the distance to the left gives the minimum of the endpoints. More analytically, we can also prove these by breaking into cases. Suppose first that $a \geq b$ so a is the right endpoint. Then $|a-b| = a-b$, and

$$\frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{a-b}{2} = a = \max(a,b),$$

while

$$\frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} - \frac{a-b}{2} = b = \min(a,b).$$

If b is the maximum and a is the minimum, then

$$\frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{b-a}{2} = b = \max(a,b),$$

while

$$\frac{a+b}{2} - \frac{|a-b|}{2} = \frac{a+b}{2} - \frac{b-a}{2} = a = \min(a,b).$$

3. Show by mathematical induction that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for all $n \geq 1$.

Solution: With $n = 1$, we have $1 = 1$, so the base case is established. Now assume the formula has been proved for $n = k$ and consider the next case $n = k+1$. We have by the induction hypothesis

$$\begin{aligned}
(1^3 + 2^3 + \cdots + k^3) + (k+1)^3 &= (1 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= (1 + 2 + \cdots + (k+1))^2
\end{aligned}$$

4. Show by mathematical induction that $n! \geq 2^n$ for all $n \geq 4$.

Solution: The base case here is $n = 4$ and $4! = 24 > 16 = 2^4$ is true. For the induction step, assume $k! > 2^k$ and consider $(k+1)!$. We see, by the induction hypothesis:

$$(k+1)! = (k+1)k! > (k+1)2^k > 2 \cdot 2^k = 2^{k+1}.$$

(The third step is valid since $k \geq 4$ implies $k+1 > 2$.)