MATH 242 – Principles of Analysis
Solutions for Problem Set 1 – due: Feb. 1

'A' Section

1. Let $A = \{x \in \mathbf{R} \mid x^2 - 5x + 6 = 0\}$, $B = (0,4) = \{x \in \mathbf{R} \mid 0 < x < 4\}$ and $C = \{\frac{x}{x^2+1} \mid x \in \mathbf{R}\}$ (Note: C is the range of the function f defined by $f(x) = \frac{x}{x^2+1}$.)
   a. Express the set C as a union of one or more closed intervals $[a,b]$ in $\mathbf{R}$. (Note: You should use facts from calculus to solve this. Don't worry that we have not justified them yet.)
   Solution: The function $f(x) = \frac{x}{x^2+1}$ has $f'(x) = \frac{1-x^2}{x^2+1}$. This is $= 0$ at $x = \pm 1$. Moreover $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ for $x > 1$. Therefore, at $x = -1$, f has a local minimum with $f(-1) = -1/2$. Similarly, at $x = 1$, f has a local maximum with $f(1) = 1/2$. We also see $\lim_{x\to\pm\infty} f(x) = 0$. Hence $f(-1) = -1/2$ is also an absolute minimum, and $f(1) = 1/2$ is also an absolute maximum. We will show later in the course that every y with $-1/2 < y < 1/2$ is also in the range. Hence $C = [-1/2, 1/2]$.
   b. Find the sets $A \cap C$ and $B \cap C$.
   Solution: Since $A = \{2,3\}$, we see that $A \cap C = \emptyset$ and $B \cap C = (0, 1/2]$.
   c. Find the sets $B \cup A$ and $B \cup C$ and express as unions of intervals in $\mathbf{R}$.
   Solution: We have $B \cup A = (0,4) = B$, since $A \subset B$. Then by part a, $B \cup C = (0,4) \cup [-1/2, 1/2] = [-1/2, 4)$.
2. Let $B_n = \{1, 1/4, 1/9, \ldots, 1/n^2\}$ for each natural number $n \geq 1$. What are $\cap_{n=1}^{\infty} B_n$ and $\cup_{n=1}^{\infty} B_n$?
   Solution: The union, $\cup_{n=1}^{\infty} B_n$, is the set
   $$\{1/n^2 \mid n \geq 1\}.$$
   The intersection, $\cap_{n=1}^{\infty} B_n$, is the set $\{1\}$, since that is the only element in B_n for all $n \geq 1$.
3. Let $I_n = [-1/n, 1/n]$ for any $n \geq 1$. What are $\cap_{n=1}^{\infty} I_n$ and $\cup_{n=1}^{\infty} I_n$. (Explain your reasoning intuitively.)
   Solution: Note first that $I_m \subset I_n$ whenever $m > n$. This shows that the union is the same as $I_1 = [-1,1]$. The intersection contains only 0. We will see in about a week how to justify the claim that for any real $a > 0$, there is some $n \geq 1$ such that $1/n < a$. Hence a is not in the intersection. The same is true on the negative side: for any $b < 0$, there exists some $n \geq 1$ such that $b < -1/n$. Hence b is not in the

intersection either. This leaves only 0 which does satisfy $-1/n < 0 < 1/n$ for all $n \geq 1$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^2 - 4x + 1$.
   a. Is f one-to-one? Why or why not?
   Solution: By completing the square, we see $x^2 - 4x + 1 = (x-2)^2 - 3$. From this we can see for instance that $f(3) = -2 = f(1)$. Therefore f is not one-to-one. We also see that the graph $y = f(x)$ is a shifted parabola with vertex at $(2, -3)$. This fact can be used to see parts of what we are saying in the later parts of the problem.
   b. Is f onto? Why or why not?
   Solution: By the same computations as for part a, we see that $f(x) \geq -3$ for all x. Therefore f is not onto $\mathbf{R}$.
   c. If $I = (1, 3)$, what is the set $f(I)$? Explain.
   Solution: f has a local and global minimum at $f(2) = -3$. Hence $f((1, 3)) = [-3, -2)$.
   d. If $J = (5, 6)$, what is the set $f^{-1}(J)$. Explain.
   Solution: We have $f(x) = x^2 - 4x + 1 = (x-2)^2 - 3 = 5$ when $x = 2 \pm \sqrt{8}$. Similarly, $f(x) = 6$ when $x = 2 \pm 3 = -1, 5$. Hence $f^{-1}(J)$ is the union of the two intervals $f^{-1}(J) = (-1, 2 - \sqrt{8}) \cup (2 + \sqrt{8}, 5)$.

'B' Section

1. Prove part (f) of Theorem 1.1.3 in the text. These are the *De Morgan Laws* for complements.
   Solution: We show $(A \cap B)^c = A^c \cup B^c$. Let $x \in (A \cap B)^c$, then $x \notin A \cap B$, which says $x \notin A$ *or* $x \notin B$. But then $x \in A^c \cup B^c$, and it follows that $(A \cap B)^c \subset A^c \cup B^c$. Conversely, if $x \in A^c \cup B^c$, then $x \notin A$ or $x \notin B$. This shows $x \notin A \cap B$, so $x \in (A \cap B)^c$, and it follows that $A^c \cup B^c \subset (A \cap B)^c$. Since we have both inclusions, $(A \cap B)^c = A^c \cup B^c$. The second statement $(A \cup B)^c = A^c \cap B^c$ is proved similarly.
2. Let A and B be arbitrary sets. Does $B = A - (A - B)$, as we might expect if we looked at the formula through the lens of ordinary algebra? If this is always true, prove it; if it is not, give both a counterexample (an example where the formula is not true), and a correct statement with proof.
   Solution: This is *not true* in general as the following counterexample shows. Let $A = \{a\}$ and let $B = \{b\}$ (with $a \neq b$). Then $A - B = \{a\} = A$, so $A - (A - B) = \emptyset \neq B$. The statement that is true here is that $A - (A - B) = A \cap B$. To prove this quickly,

the best way is probably to use the De Morgan Laws from question 1 and other parts of Theorem 1.1.3 in the text. We have

$$\begin{aligned} A-(A-B) &= A\cap(A-B)^c \\ &= A\cap(A\cap B^c)^c \\ &= A\cap(A^c\cup(B^c)^c) \qquad \text{(by 1.1.3 (f))} \\ &= A\cap(A^c\cup B) \qquad \text{(by 1.1.3 (a))} \\ &= (A\cap A^c)\cup(A\cap B) \qquad \text{(by 1.1.3 (e))} \\ &= \emptyset\cup(A\cap B) \\ &= A\cap B. \end{aligned}$$

3. Let $f : A \to B$ be a function.
   a. Let C, D be subsets of A. Is it always true that $f(C\cup D) = f(C)\cup f(D)$? If this is always true prove it; if it is not, give a counterexample.
   Solution: This statement is always true. We can prove it as follows. If $x \in C\cup D$, then $x \in C$ or $x \in D$. Hence $f(x) \in f(C)$ or $f(x) \in f(D)$. It follows that $f(x) \in f(C)\cup f(D)$, so $f(C\cup D) \subset f(C)\cup f(D)$. Conversely, if $y \in f(C)\cup f(D)$, then $y \in f(C)$ or $y \in f(D)$. So $y = f(x)$ for some $x \in C$ or $y = f(x)$ for some $x \in D$. It follows that $y \in f(C\cup D)$, so $f(C)\cup f(D) \subset f(C\cup D)$. This shows the equality.
   b. Show that f is onto if and only if $f(f^{-1}(E)) = E$ for all subsets E of B.
   Solution: Suppose that $f(f^{-1}(E)) = E$ for all subsets $E \subset B$. Let $b \in B$ and $E = \{b\}$, then $f^{-1}(E) \neq \emptyset$ since $f(f^{-1}(E)) = E$. Thus there is some $a \in f^{-1}(E)$, so $f(a) = b$. Since this is true for all $b \in B$, f is onto. Conversely, if f is onto, we must show $f(f^{-1}(E)) = E$ for all subsets $E \subset B$. So let E be an arbitrary subset of B. The definition of the inverse image says $f(f^{-1}(E)) \subset E$ for all mappings f (that is, even without the assumption that f is onto). If in addition we know that f is onto, we have that for all $b \in E$, there exist $a \in A$ such that $f(a) = b$, and hence that those $a \in f^{-1}(E)$. It follows that $E \subset f(f^{-1}(E))$ when f is onto. Hence if f is onto, then $f(f^{-1}(E)) = E$.
4. Let $f : A \to B$ and $g : B \to C$.
   a. Show that if f and g are both one-to-one, then $g\circ f : A \to C$ is also one-to-one.
   Solution: Let $(g\circ f)(x) = (g\circ f)(y)$ for some $x, y \in A$. Then $g(f(x)) = g(f(y))$. Since g is assumed to be one-to-one, we have $f(x) = f(y)$. But then, because f is assumed to be one-to-one, $x = y$. Therefore, $g\circ f$ is one-to-one.

b. Is the converse of the statement in part a true? That is, if you know that $g \circ f$ is one-to-one, does it follow that f and g are one-to-one? Prove or find a counterexample.

Solution: This statement is *not true*. For instance, consider $f : \{b, c\} \to \{b, c\}$ defined by $f(b) = b$ and $f(c) = c$. Also let $g : \{a, b, c\} \to \{b, c\}$ by $g(a) = g(b) = b$ and $g(c) = c$. Then $g \circ f : \{b, c\} \to \{b, c\}$ satisfies $(g \circ f)(b) = b$ and $(g \circ f)(c) = c$ so $g \circ f$ is one-to-one. However, g is not one-to-one. (The statement that is true here is that f must be one-to-one and g must be one-to-one when restricted to the range of f.)