Mathematics 242 – Principles of Analysis
Solutions for Midterm Exam 3
May 3, 2013

*Directions*

Do all work in the blue exam booklet. There are 100 possible regular points and 10 possible Extra Credit points. Possibly useful information:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

I. Both parts of this question refer to the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$.

A) (25) Show directly, using upper and lower sums, that $f$ is integrable on $[0, 1]$ and determine the value $\int_0^1 x^2 \, dx$.

*Solution:* Consider a regular partition $\mathcal{P}_n$ of $[0, 1]$ with $n$ smaller intervals. Then $\Delta x = \frac{1}{n}$ and the endpoints are $x_i = i/n$ for $i = 0, \ldots, n$. Since $x^2$ is increasing on the interval $[0, 1]$ we have $m_i = ((i-1)/n)^2$ and $M_i = (i/n)^2$, so

$$\begin{aligned} L(f, \mathcal{P}_n) &= \sum_{i=1}^{n} ((i-1)/n)^2 (1/n) \\ &= \frac{(n-1)n(2n-1)}{6n^3}, \text{ and} \\ U(f, \mathcal{P}_n) &= \sum_{i=1}^{n} (i/n)^2 (1/n) \\ &= \frac{n(n+1)(2n+1)}{6n^3}. \end{aligned}$$

Hence

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{1}{n}$$

Given $\varepsilon > 0$, the difference will be $< \varepsilon$ whenever $n > \frac{1}{\varepsilon}$. So the function is integrable.

*Note:* The same result can be derived using the observation we made in the proof of the general result that monotone functions are integrable. Then $U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = \frac{(f(b)-f(a))(b-a)}{n} = \frac{1}{n}$. This is certainly "cleaner!")

The value of the integral is

$$\lim_{n\to\infty} U(f, \mathcal{P}_n) = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{1}{3}.$$

B) (5) Compute $\int_0^1 x^2\ dx$ using the Fundamental Theorem of Calculus to check your work.

*Solution:* An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$, so by part (2) of the FTC, the value is

$$\int_0^1 x^2\ dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}.$$

II. Let $f(x) = \frac{\sin(x)}{x}$ for $x \neq 0$.

A) (5) How should a value $f(0)$ be defined to make $f$ continuous at $x = 0$? (Note: If you do not how to decide, you may "buy" the answer for 5 points so that you can do the next part.)

*Solution:* By results from Problem Set 7 and 8, we know

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

so we should set $f(0) = 1$ to get continuity. (Note that $\frac{\sin(x)}{x}$ is differentiable, hence continuous at all $x \neq 0$.)

B) (15) Using your value from part A for $f(0)$, use the limit definition to show that the resulting function is differentiable and find the value $f'(0)$. Say what method you are using to compute this limit.

*Solution:* We need to compute

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\sin(x)}{x} - 1}{x - 0} = \lim_{x\to 0} \frac{\sin(x) - x}{x^2}.$$

This is a 0/0 indeterminate form, so we can apply L'Hopital's Rule (twice) as follows:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{x^2} &= \lim_{x\to 0} \frac{\cos(x) - 1}{2x} \text{ (still 0/0)} \\
&= \lim_{x\to 0} \frac{-\sin(x)}{2} \\
&= 0
\end{aligned}$$

So $f'(0)$ exists and equals 0. (The graph $y = f(x)$ has a local maximum at $x = 0$.)

III.

A) (10) State the Mean Value Theorem.

*Solution:* Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

B) Determine

$$\text{lub}\left\{\frac{\ln(x)}{x} \mid x \in (e, +\infty)\right\}$$

and justify your answer completely using facts we have developed in this course.

*Solution:* Let $f(x) = \frac{\ln(x)}{x}$. By the quotient rule, we have $f'(x) = \frac{1-\ln(x)}{x^2}$. Therefore, $f'(x) > 0$ if $x < e$ and $f'(x) < 0$ for $x > e$. By one of our corollaries of the MVT, $f$ is strictly decreasing on the given interval. That means that the value $f(e) = \frac{1}{e}$ at the left endpoint is the least upper bound of the set of values.

*Note:* It is not enough just to notice that $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$. To say the lub of the set of values equals $f(e)$, you must also show that $f$ does not get larger anywhere between $e$ and $\infty$. Showing $f$ is strictly decreasing implies that.

IV. True-False. Say whether each of the following statements is true or false. For true statements, give short proofs or reasons; for false ones give reasons or counterexamples.

A) (10) $f(x) = \ln(x)$ is uniformly continuous on $(1, +\infty)$.

*Solution:* TRUE. $f'(x) = \frac{1}{x}$ satisfies $|f'(c)| < 1$ for all $c \in (1, +\infty)$. Therefore, as we saw on PS 8, $f$ must be uniformly continuous on the interval $(1, +\infty)$. Here's the proof again: The MVT implies that if $1 < x_1 < x_2$, then there is some $c \in (x_1, x_2)$ where

$$|f(x_2) - f(x_1)| = |f'(c)||x_2 - x_1|.$$

But $|f'(c)| < 1$, so this shows

$$|f(x_2) - f(x_1)| < |x_2 - x_1|.$$

Hence $|f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \delta = \varepsilon$. (Or alternatively, the bound on $|f'(x)|$ shows that $f$ is Lipschitz continuous on that interval, with Lipschitz constant $k = 1$.)

B) (10) If $A \subset \mathbf{R}$ contains no nonempty interval $(a, b)$, then $A$ is finite or countably infinite.

*Solution:* FALSE. The Cantor set is a counterexample. It contains no intervals since it is a subset of $[0, 1]$, but the intervals removed in its construction have total length 1. On the other hand, we saw in class that the Cantor set is uncountably infinite.

C) (10) Let $f$ be continuous on $[a, b]$ and assume $f(x) > 1$ for all $x \in [a, b]$. It is possible for $\text{glb}\{f(x) \mid x \in [a, b]\}$ to equal 1.

*Solution:* FALSE. By the EVT, we have that $f$ attains a minimum $m = f(c)$, for some $c \in [a, b]$. This $m$ must equal the greatest lower bound of the set of values of $f$. By the given information $f(c) > 1$. Therefore the glb must also be strictly larger than 1.

*Extra Credit* (10) Show that if $f$ is differentiable on an open interval $I$ and $[a, b] \subset I$ where $f'(a) < 0$ and $f'(b) > 0$, then there must be some $c \in (a, b)$ where $f'(c) = 0$.

*Solution:* Since differentiability of $f$ implies continuity of $f$, the EVT applies to $f$ on the closed interval $[a, b]$ and $f$ attains a minimum value at some $c \in [a, b]$. We claim first that $c \neq a, b$. Arguing by contradiction, suppose $f(a) \leq f(x)$ for all $x \in [a, b]$. Then we have $\frac{f(x)-f(a)}{x-a} \geq 0$ for all $x \in [a, b]$, and this implies

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \geq 0.$$

But this contradicts the given information that $f'(a) < 0$. Hence the minimum is not attained at $a$. A similar argument shows the minimum is not attained at $b$ either. Hence it must be attained at some $c \in (a, b)$. But now by the usual argument, we have

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0,$$

while

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

So $f'(c) = 0$.

*Notes:*

1) You cannot apply Rolle's theorem directly here because we are not given that $f(a) = f(b)$.
2) In addition, it is not correct to assume (even without saying so!) that $f'$ is continuous, since that is not given either. This means, for instance, that the IVT does not necessarily apply to $f'$ here.
3) This is the main step in a proof of a result called *Darboux's Theorem*, which says essentially that derivatives attain intermediate values, just as continuous functions do by the IVT. The general statement says for instance that if $f'(a) < k < f'(b)$ then there is some $c \in (a, b)$ with $f'(c) = k$. We can see that by applying the result here to the function $g(x) = f(x) - kx$. It is not necessarily true that the derivative function $f'(x)$ is continuous, though, so this fact is *not a consequence of the IVT.*