Mathematics 242 – Principles of Analysis
Solutions for Exam 1 – February 22, 2013

I.

A) (10) Let

$$A = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2 - \frac{1}{n}\right)$$

Explain why A is bounded and determine upper and lower bounds for A.

Solution: A is bounded below since $\frac{1}{n} > 0$ for all n. Therefore all elements of A are strictly positive, or $x > 0$ for all $x \in A$. On the other hand $2 - \frac{1}{n} < 2$ for all n, so all elements of A satisfy $x < 2$.

B) (10) *Define:* The real number a is a *least upper bound* of $A \subset \mathbf{R}$, and state the Least Upper Bound Axiom for $\mathbf{R}$.

Solution: a is a least upper bound of A if (1) $a \geq x$ for all $x \in A$, and (2) If $b \geq x$ for all $x \in A$, then $b \geq a$. The LUB Axiom states that every nonempty set of real numbers that is bounded above has a least upper bound in $\mathbf{R}$. (The existence is not guaranteed just by defining a term by giving the properties that a lub should satisfy.)

C) (10) Let A be a bounded subset of $\mathbf{R}$ and let $B = \{4 \cdot x - 3 \mid x \in A\}$. What can be said about lub(B)? Prove your assertion.

Solution: If $a = \text{lub}(A)$, then we claim $\text{lub}(B) = 4a - 3$. To prove this, note that a exists in $\mathbf{R}$ by the LUB Axiom. Since $a \geq x$ for all $x \in A$, we have $4a \geq 4x$ and $4a - 3 \geq 4x - 3$ by properties of the order relation. This shows that $4a - 3$ is an upper bound for B. Then, if b is any upper bound for B, we have $b \geq 4x - 3$ for all $x \in A$, so $\frac{b+3}{4} \geq x$ for all $x \in A$. This implies $\frac{b+3}{4} \geq a$ by definition of an lub. Hence $b \geq 4a - 3$. This shows that $4a - 3 = 4\text{lub}(A) - 3 = \text{lub}(B)$.

II. (20) Let x_n be the sequence defined by the rules $x_1 = 1$ and $x_{n+1} = \frac{1}{3}x_n + 1$ for all $n \geq 1$. Show by mathematical induction that

$$x_n = \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} \text{ for all } n \geq 1.$$

Solution: The base case is $n = 1$. We have $x_1 = 1 = \frac{1-\frac{1}{3}}{1-\frac{1}{3}}$, so the formula is true in this case. Now assume that

$$x_k = \frac{1 - \frac{1}{3^k}}{1 - \frac{1}{3}}.$$

By definition,

$$\begin{aligned}
x_{k+1} &= \frac{1}{3}x_k + 1 \quad \text{(by definition of the sequence)} \\
&= \frac{1}{3} \cdot \frac{1 - \frac{1}{3^k}}{1 - \frac{1}{3}} + 1 \quad \text{(by the induction hypothesis)} \\
&= \frac{\frac{1}{3} - \frac{1}{3^{k+1}}}{1 - \frac{1}{3}} + 1 \\
&= \frac{\frac{1}{3} - \frac{1}{3^{k+1}} + 1 - \frac{1}{3}}{1 - \frac{1}{3}} \quad \text{(common denominator)} \\
&= \frac{1 - \frac{1}{3^{k+1}}}{1 - \frac{1}{3}} \quad \text{(by algebra)}
\end{aligned}$$

This shows that x_n is given by the formula above for all $n \geq 1$.

III. Let $x_n = \frac{5^n}{7^n+3}$ for all natural numbers $n \geq 1$.

A) (10) Determine $\lim_{n\to\infty} x_n$ intuitively.
Solution: We have

$$\frac{5^n}{7^n + 3} = \frac{(5/7)^n}{1 + 3(1/7)^n}$$

Since $0 < 5/7 < 1$ and $0 < 1/7 < 1$, $(5/7)^n \to 0$ and $(1/7)^n \to 0$. We expect the limit should be 0.

B) (20) Use the ε, n_0 definition of convergence to prove that $\{x_n\}$ converges to the number you identified in part A.
Solution: Given $\varepsilon > 0$, let n_0 be any natural number satisfying $n_0 > \frac{\ln(\varepsilon)}{\ln(5/7)}$. (Note: If $\varepsilon < 1$, then both the top and the bottom of the quotient are negative so the lower bound on n_0 is positive, and the smaller ε is the larger n_0 will be.) The for all $n \geq n_0$ we will have

$$\begin{aligned}
\left| \frac{5^n}{7^n + 3} - 0 \right| &= \frac{5^n}{7^n + 3} \\
&< \left(\frac{5}{7}\right)^n \\
&< \left(\frac{5}{7}\right)^{n_0} \quad \text{(since } 5/7 < 1 \text{ and } n \geq n_0\text{)} \\
&< \left(\frac{5}{7}\right)^{\frac{\ln(\varepsilon)}{\ln(5/7)}} \\
&= e^{\ln(5/7)\cdot\ln(\varepsilon)/\ln(5/7)} \\
&= e^{\ln(\varepsilon)} \\
&= \varepsilon.
\end{aligned}$$

This shows that $x_n \to 0$.

IV. True-False. For each true statement, give a short proof or reason. For each false statement give an explicit counterexample.

A) (10) Let A be a bounded set of real numbers and let $a = \text{lub}(A)$. Then for each $\varepsilon > 0$, there exists $x \in A$ such that $a - \varepsilon < x < a$.
Solution: This is FALSE. For example let $A = [0, 1] \cup \{2\}$. Then $a = \text{lub}(A) = 2$. But if $\varepsilon < 1$, then there are no elements of A in the open interval $(2 - \varepsilon, 2)$. (The statement would be true if the inequalities were $a - \varepsilon < x \leq a$, but it is false if we do not allow $x = a$.)

B) (10) If r is a nonzero rational number and s is a nonzero irrational number, then r/s is irrational.
Solution: This is TRUE. Since $r \neq 0$ and $s \neq 0$, $r/s = t \neq 0$. Suppose t is rational. Then we can solve to get $s = r/t \in \mathbf{Q}$, which contradicts the assumption $s \notin \mathbf{Q}$. Hence t must be irrational.

Extra Credit (10) Is it possible to produce a sequence x_n whose terms include all the rational numbers p/q with $p, q \in \mathbf{Z}$ and $p, q > 0$? If so, give an indication how to construct such a sequence. If not, give a reason why there cannot exist such a sequence.

Solution: There is such a sequence, and we can construct one as follows. List all the ratios of positive integers in a two-dimensional table with denominators constant across the rows and numerators constant along the columns:

$$\begin{array}{ccccc} 1/1 & 2/1 & 3/1 & 4/1 & \cdots \\ 1/2 & 2/2 & 3/2 & 4/2 & \cdots \\ 1/3 & 2/3 & 3/3 & 4/3 & \cdots \\ 1/4 & 2/4 & 3/4 & 4/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can then construct a sequence by starting at the upper left and listing the elements in this "zig-zag order" (down the first diagonal, up the second, and and alternating that way forever):

$$x_1 = 1/1, x_2 = 2/1, x_3 = 1/2, x_4 = 1/3, x_5 = 2/2, x_6 = 3/1, x_7 = 4/1, x_8 = 3/2, \ldots$$

Every positive rational will appear somewhere in this sequence (in fact infinitely many times each, do you see why?)