Mathematics 242, section 1 – Principles of Analysis
Solutions For Final Examination – May 15, 2013

I. (20) Let $A = \{x^2 - 2 : -1 < x < 2\}$ and $B = \{x : |x - 1| < 4\}$. Find $\text{lub}(A \cup B)$ and $\text{glb}(A \cap B)$.

Solution: By the definitions, we see $A = [-2, 2)$ and $B = (-3, 5)$. Hence, since $A \subset B$, $A \cup B = B = (-3, 5)$ and $A \cap B = A = [-2, 2)$. Hence $\text{lub}(A \cup B) = 5$ and $\text{glb}(A \cap B) = -2$.

II.

A) (10) State the ε, n_0 definition of convergence for a sequence of real numbers.

Solution: The sequence x_n converges to c if for all $\varepsilon > 0$ there exist $n_0 \in \mathbf{N}$ such that $|x_n - c| < \varepsilon$ for all $n \geq n_0$.

B) (10) Identify $\lim_{n\to\infty} \frac{5n^2+1}{n^2+n+4}$.

Solution: Using the limit theorems, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{5n^2+1}{n^2+n+4} &= \lim_{n\to\infty} \frac{(5n^2+1)\cdot\frac{1}{n^2}}{(n^2+n+4)\cdot\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{5+\frac{1}{n^2}}{1+\frac{1}{n}+\frac{4}{n^2}} \\
&= \frac{5+\lim_{n\to\infty}\frac{1}{n^2}}{1+\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{4}{n^2}} \\
&= \frac{5+0}{1+0+0} \\
&= 5.
\end{aligned}$$

C) (10) Show that your result in part B is correct using the definition.

Solution: Given $\varepsilon > 0$, let $n_0 > \max(20, \frac{6}{\varepsilon})$. (No matter how small ε is, this is possible because $\mathbf{N}$ is not bounded above.) Then for all $n \geq n_0$, we have

$$\left|\frac{5n^2+1}{n^2+n+4} - 5\right| = \left|\frac{-5n-19}{n^2+n+4}\right| < \frac{5n+19}{n^2}$$

Since $n \geq 20$, we have $5n + 19 < 6n$, so this is

$$< \frac{6n}{n^2} = \frac{6}{n} \leq \frac{6}{n_0} < \varepsilon.$$

Hence by the definition, $\frac{5n^2+1}{n^2+n+4}$ converges to 5.

III.

A) (15) Show that if x_n is a monotone increasing sequence of real numbers that is bounded above, then x_n converges to some real number.

Solution: The set $X = \{x_n\}$ is bounded above in $\mathbf{R}$. Hence the LUB axiom implies that it has a least upper bound c. Given any $\varepsilon > 0$, $c - \varepsilon$ is not an unpper bound for X, therefore there exists some n_0 such that $c - \varepsilon < x_{n_0} \le c$. But the sequence is monotone increasing so $c - \varepsilon < x_{n_0} \le x_n \le c$ for all $n \ge n_0$. That implies $|x_n - c| < \varepsilon$ for all $n \ge n_0$, and hence $x_n \to c$ by the definition.

B) (10) True/False and prove/give a reason: The sequence

$$x_n = \begin{cases} -1 + \frac{1}{n^2} & \text{if } n \text{ is a prime integer} > 1 \\ 1 - \frac{1}{n^2} & \text{if } n \text{ is not prime} \end{cases}$$

has convergent subsequences.

Solution: This is TRUE. From the definition, $x_n < 1$ for all n and $x_n > -1$ for all n. Since $\{x_n\}$ is a bounded sequence, the Bolzano-Weierstrass theorem implies it must have convergent subsequences. (In fact, we can also see directly from the above that the subsequences for $n_k = k$th prime number and $n_k = 2k$, $k \ge 1$ are convergent.)

C) True/False and prove/give a reason: The infinite "continued radical"

$$\sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}}}$$

represents a finite real number. (Hint: If so, that number would be the limit of a sequence defined by $x_1 = \sqrt{3}$ and $x_n = \sqrt{3 + x_{n-1}}$ for all $n \ge 2$.)

Solution: This is TRUE. First we notice that in the sequence described in the Hint,

$$x_2 = \sqrt{3 + \sqrt{3}}$$
$$x_3 = \sqrt{3 + \sqrt{3 + \sqrt{3}}},$$

etc. So as $n \to \infty$, if the sequence x_n converges, it converges to the expression defined by the infinite "continued radical." We claim next the sequence is bounded below by 0 (clear) and above by 3. The second part requires some proof, but we can argue that $x_n < 3$ for all n by induction as follows. The base case is $x_1 = \sqrt{3} < 3$. Assuming $x_k < 3$, then we have $x_{k+1} = \sqrt{3 + x_k} < \sqrt{3 + 3} = \sqrt{6} < 3$. Hence $x_n < 3$ for all n. Finally, we show by induction again that x_n is monotone increasing. Note that $x_2 = \sqrt{3 + \sqrt{3}} > \sqrt{3} = x_1$ since $x_2^2 = 3 + \sqrt{3} > 3 = x_1^2$. Assuming $x_{k+1} > x_k$, then

$x_{k+2} = \sqrt{3 + x_{k+1}} > \sqrt{3 + x_k} = x_{k+1}$ also. Hence the monotone convergence theorem (part A) implies that $x_n \to a$ for some real number $a < 3$. In fact, taking the limit on both sides of $x_{n+1} = \sqrt{3 + x_n}$, we can see that a must satisfy

$$a = \sqrt{3 + a} \Rightarrow a^2 - a - 3 = 0$$

By the quadratic formula,

$$a = \frac{1 + \sqrt{13}}{2} \doteq 2.302775638.$$

IV.

A) (15) Let

$$f(x) = \begin{cases} x + 3 & \text{if } x \text{ is a rational number} \\ -x^2 + 3 & \text{if } x \text{ is an irrational number} \end{cases}$$

Is f continuous at $x = 0$? Why or why not?

Solution: Yes, $f(x)$ is continuous at 0. To see this, let $\varepsilon > 0$, and $\delta < \min(\varepsilon, \sqrt{\varepsilon})$. If $|x| < \delta$ and x is rational then we have

$$|f(x) - f(0)| = |x + 3 - 3| = |x| < \varepsilon,$$

while if x is irrational we have

$$|f(x) - f(0)| = |-x^2 + 3 - 3| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

Therefore f is continuous at 0.

B) (25) State and prove the Intermediate Value Theorem. (You may assume as known the theorem that if f is continuous at c and $x_n \to c$ is a sequence contained in the domain of f, then $f(x_n) \to f(c)$).

Solution: The IVT in the form we proved first says that if f is continuous on $[a, b]$ with $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = k$. We proved this as follows. If $k = f(a)$ or $k = f(b)$, then there is nothing to prove. Hence we must consider the case that k is strictly between the two values $f(a)$ and $f(b)$. Suppose, for instance, that $f(a) < k < f(b)$. (The other case $f(b) < k < f(a)$ can be proved similarly.) Consider

$$S = \{x \in [a, b] \mid f(x) \leq k\}$$

Then S is not empty since for instance $a \in S$. S is also bounded since it is a subset of $[a, b]$. Hence the LUB axiom implies that $c = \text{lub}(S)$ exists in $\mathbf{R}$. We claim that $f(c) = k$. First note that $c < b$ since $f(b) > k$. Hence if we let $x_n \to c$ with $c \leq x_n \leq b$, we will have $f(x_n) > k$ for all n, and hence (since f is continuous at c,

$\lim_{n\to\infty} f(x_n) = f(c) \geq k$. On the other hand, since $c = \text{lub}(S)$, if $\varepsilon > 0$, then there will exist $x \in S$ with $c - \varepsilon < x \leq c$. Apply this for each $\varepsilon = \frac{1}{n}$ for $n \in \mathbf{N}$. We get a a sequence $x'_n \in S$ with $x'_n \to c$. Hence $\lim_{n\to\infty} f(x'_n) = f(c) \leq k$ since $f(x'_n) \leq k$ for all n. The two inequalities show that $f(c) = k$.

V.

A) (15) Using the limit definition of the derivative, compute $f'(c)$ for $f(x) = \frac{1}{(x+3)^2}$ at a general $c \neq -3$.

Solution: We have

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x\to c} \frac{\frac{1}{(x+3)^2} - \frac{1}{(c+3)^2}}{x - c} \\ &= \lim_{x\to c} \frac{(c+3)^2 - (x+3)^2}{(x+3)^2(c+3)^2(x-c)} \\ &= \lim_{x\to c} \frac{c^2 - x^2 + 6(c-x)}{(x+3)^2(c+3)^2(x-c)} \\ &= \lim_{x\to c} \frac{-c - x - 6}{(x+3)^2(c+3)^2} \\ &= \frac{-2}{(c+3)^3} \end{aligned}$$

(*Note:* It is not permissible to use L'Hopital's Rule to compute this limit because that requires you to use a derivative formula to derive $f'(x) = \frac{-2}{(x+3)^3}$. Using the derivative rule to compute the derivative by the definition is an example of circular reasoning!)

B) (10) What theorem guarantees that

$$F(x) = \int_1^x \frac{1}{(t+3)^2}\ dt$$

is differentiable at $x = 2$? Exactly why does it apply here? What does it say about $F'(2)$?

Solution: The theorem is the first part of the Fundamental Theorem of Calculus. It applies because the function $f(x) = \frac{1}{(x+3)^2}$ is continuous on the closed interval $[1, a]$ for all $a > 2$. It implies $F'(2) = f(2) = \frac{1}{25}$.

C) (10) Show that if $f(x) = \frac{e^x + e^{-x}}{2}$, then for every real k, there exists a solution c of the equation $f'(c) = \frac{e^c - e^{-c}}{2} = k$.

Solution: Note that $f'(x) = \frac{e^x - e^{-x}}{2}$ exists and is continuous at all $x \in \mathbf{R}$. This function satisfies $\lim_{x\to+\infty} f'(x) = +\infty$ and $\lim_{x\to-\infty} f'(x) = -\infty$. So whatever the value k, we can find an interval $[-a, a]$ such that $f'(-a) < k < f'(a)$. On the interval $[-a, a]$, f' is continuous, and then the Intermediate Value Theorem implies there is some $c \in [-a, a]$ such that $f'(c) = k$.

Alternatively, it would also be possible to solve the equation $\frac{e^c - e^{-c}}{2} = k$ explicitly like this:

$$\begin{aligned}
\frac{e^c - e^{-c}}{2} = k &\Leftrightarrow e^{2c} - 2ke^c - 1 = 0 \\
&\Leftrightarrow e^c = \frac{2k + \sqrt{4k^2 + 4}}{2} \quad \text{(by the quadratic formula)} \\
&\Leftrightarrow e^c = k + \sqrt{k^2 + 1} \\
&\Leftrightarrow c = \ln(k + \sqrt{k^2 + 1})
\end{aligned}$$

The function $g(x) = \ln(x + \sqrt{x^2 + 1})$ is also known as the inverse hyperbolic sine function, since $f'(x) = \frac{e^x - e^{-x}}{2}$ is the hyperbolic sine.

VI. (20) In this question, you may use the summation formulas:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Show that $f(x) = x^2 + x$ is integrable on $[a, b] = [0, 2]$ by considering upper and lower sums for f and determine the value of $\int_0^2 x^2 + x\, dx$.

Solution: $f'(x) = 2x + 1 > 0$ for all $x \in [0, 2]$. Therefore, f is increasing on $[0, 2]$ and as in the proof of the integrability of all monotone functions, if $\mathcal{P}_n$ is the regular partition of $[0, 2]$ into n equal subintervals,

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) = (f(2) - f(0))\frac{(2-0)}{n} = \frac{12}{n}.$$

If $\varepsilon > 0$ is given and $n > \frac{12}{\varepsilon}$, then

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) < \varepsilon.$$

The limit of the upper sum for $\mathcal{P}_n$ as $n \to \infty$ is

$$\begin{aligned}
\int_0^2 x^2 + x\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^2 + \frac{2i}{n} \right) \frac{2}{n} \\
&= \lim_{n\to\infty} \left(\frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{4}{n^2} \sum_{i=1}^{n} i \right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \right) \\
&= \frac{8}{3} + 2 = \frac{14}{3}.
\end{aligned}$$

Check: By Part 2 of the FTC:

$$\int_0^2 x^2 + x\ dx = \left.\frac{x^3}{3} + \frac{x^2}{2}\right|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}.$$

VII.

A) (5) State the definition of convergence for an infinite series $\sum_{n=1}^{\infty} a_n$.

Solution: The infinite series $\sum_{n=1}^{\infty} a_n$ converges to $S \in \mathbf{R}$ if the sequence of partial sums $s_k = \sum_{n=1}^{k} a_n$ converges to S (as a sequence).

B) (15) For which $x \in \mathbf{R}$ does the power series

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n 7^n}$$

converge? (Apply the Ratio Test and test the series at the endpoints of your interval separately.)

Solution: We have

$$\lim_{n\to\infty} \left| \frac{3^{n+1}x^{n+1}}{(n+1)7^{n+1}} \cdot \frac{n7^n}{3^n x^n} \right| = \lim_{n\to\infty} \frac{3n}{7(n+1)}|x| = \frac{3|x|}{7}.$$

By the Ratio Test, the series converges absolutely whenever $\frac{3|x|}{7} < 1$, or $|x| < \frac{7}{3}$, and it diverges when $|x| > \frac{7}{3}$. At $x = \pm\frac{7}{3}$, we have the following. If $x = \frac{7}{3}$, we substitute and simplify to obtain $\sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, which diverges. If $x = -\frac{7}{3}$, we substitute and simplify to obtain $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This series converges by the Alternating Series Test. Therefore the series converges at all $x \in \left[-\frac{7}{3}, \frac{7}{3}\right)$ (and only for those x).

Extra Credit (20) Prove the result mentioned in part B of question IV above: if f is continuous at c and $x_n \to c$ is a sequence contained in the domain of f, then $f(x_n) \to f(c)$.

Solution: Since f is continuous at c, given any $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \tag{1}$$

for all x with $|x - c| < \delta$. Then, since $x_n \to c$, given that δ, there is an n_0 such that $|x_n - c| < \delta$ for all $n \geq n_0$. But then substituting $x = x_n$ from the sequence into the inequality (1) we get $|f(x_n) - f(c)| < \varepsilon$ for all $n \geq n_0$. This implies that the sequence $f(x_n) \to f(c)$.