Mathematics 242 – Principles of Analysis
Solutions for Midterm Exam 3
May 6, 2011

I. Both parts of this question refer to the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 - 6x + 3$.

A) (20) Consider the regular partitions $\mathcal{P}_n$ of the interval $[0, 2]$ and show directly, using the upper and lower sums, that f is integrable on $[0, 2]$.

Solution: Note that f is *decreasing* on $[0, 2]$ since $f'(x) = 2x - 6 < 0$ for all x with $0 \le x \le 2$. The partition is

$$\mathcal{P}_n = \{0, 2/n, 4/n, \ldots, 2\},$$

with $x_i = 2i/n$ for $i = 0, 1, \ldots, n$. Hence, since f is smallest at the right endpoint in each subinterval,

$$\begin{aligned} L_{\mathcal{P}_n}(f) &= \sum_{i=1}^{n} \left((2i/n)^2 - 6 \cdot (2i/n) + 3\right) \frac{2}{n} \\ &= \frac{8}{n^3} \sum_{i=1}^{n} i^2 - \frac{24}{n^2} \sum_{i=1}^{n} i + \frac{6}{n} \sum_{i=1}^{n} 1 \\ &= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{24}{n^2} \cdot \frac{n(n+1)}{2} + 6 \\ &= \frac{4}{3} + \frac{4}{n} + \frac{4}{3n^2} - 12 - \frac{12}{n} + 6 \\ &= \frac{-10}{3} - \frac{8}{n} + \frac{4}{3n^2} \end{aligned}$$

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Similarly, f is largest at the left endpoint in each subinterval, so

$$\begin{aligned} U_{\mathcal{P}_n}(f) &= \sum_{i=1}^{n} \left((2(i-1)/n)^2 - 6 \cdot (2(i-1)/n) + 3\right) \frac{2}{n} \\ &= \frac{8}{n^3} \sum_{i=1}^{n-1} i^2 - \frac{24}{n^2} \sum_{i=1}^{n-1} i + \frac{6}{n} \sum_{i=1}^{n} 1 \\ &= \frac{8}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} - \frac{24}{n^2} \cdot \frac{(n-1)n}{2} + 6 \\ &= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} - 12 + \frac{12}{n} + 6 \\ &= \frac{-10}{3} + \frac{8}{n} + \frac{4}{3n^2} \end{aligned}$$

.

Therefore, for any given $\varepsilon > 0$, if $n > 16/\varepsilon$,

$$U_{\mathcal{P}_n}(f) - L_{\mathcal{P}_n}(f) = \frac{16}{n} < \varepsilon.$$

This shows that f is integrable.

B) (15) Explain why the hypothesis of the Mean Value Theorem is satisfied for f on the interval $[1, 5]$ and find the number c mentioned in the conclusion.
Solution: f is a polynomial function, so it is differentiable, hence continuous everywhere. On the interval $[1, 5]$, $f(5) - f(1) = -2 - (-2) = 0$. The MVT says that there is some $c \in (1, 5)$ where $f'(c) = 0 \cdot (5 - 1) = 0$. Since $f'(c) = 2c - 6 = 0$, this is true for $c = 3$.

II. (15) For which $a \geq 0$ is

$$f(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f continuous at $x = 0$? For which $a \geq 0$ is f differentiable at $x = 0$?

Solution: Note that this function is not defined for $x < 0$ for some a. Hence in the limits below, we will consider the case $x \to 0^+$ only. When $x \to 0^-$ also makes sense, the limits will be the same. For continuity at $x = 0$, we must have $\lim_{x\to 0^+} x^a \sin(1/x) = f(0) = 0$. This will be true by the squeeze theorem for limits as long as $a > 0$:

$$x^a \cdot (-1) \leq x^a \sin(1/x) \leq x^a \cdot 1$$

for all $x > 0$. Hence since $\lim_{x\to 0^+} x^a = 0$ if $a > 0$,

$$\lim_{x\to 0^+} x^a \sin(1/x) = 0 = f(0).$$

This is not true with $a = 0$ since then the limit does not exist. For differentiability at $x = 0$, the limit

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} x^{a-1} \sin(1/x)$$

must exist. This will be true (and the limit will equal zero) as long as $a > 1$ (squeeze theorem again).

III. (20) Show that if f is monotone increasing on $[a, b]$, then f is integrable on $[a, b]$.

Solution: If f is monotone increasing on $[a, b]$ and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ is a regular partition, then

$$U_{\mathcal{P}}(f) - L_{\mathcal{P}}(f) = \sum_{i=1} f(x_i)\frac{b-a}{n} - \sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n}.$$

This is a telescoping sum, which cancels to

$$\frac{(f(b) - f(a))(b - a)}{n}.$$

Given $\varepsilon > 0$, this difference will be $< \varepsilon$ as soon as $n > \frac{(f(b)-f(a))(b-a)}{\varepsilon}$. Therefore f is integrable on $[a, b]$.

IV. True-False. Say whether each of the following statements is true or false. For true statements, give short proofs; for false ones give reasons or counterexamples. Do any *three* parts. If you submit solutions for all four, then I will consider the other one for Extra Credit.

A) (10) Let $f(x) = e^x - e^{2x}$. There exists some $c \in (0, \ln(3))$ such that $f(c) = -2$.

   Solution: The statement is TRUE. We apply the IVT. First, f is continuous everywhere since the exponentials e^x and e^{2x} are differentiable everywhere. On the interval $[0, \ln(3)]$, we have $f(0) = 1 - 1 = 0$ and $f(\ln(3)) = 3 - 9 = -6$. Since -2 is in the range between the endpoint values, the ("weak form" of) the IVT implies that there exists $c \in (0, \ln(3))$ such that $f(c) = -2$.

B) (10) If f is differentiable on (a, b) and there exists a real number $B \geq 0$ such that $|f'(x)| < B$ for all $x \in (a, b)$, then f is uniformly continuous on (a, b).

   Solution: This is TRUE. We apply the MVT to f on the interval $[x, x']$ where $a < x < x' < b$ are arbitary. Then there exists a $c \in (x, x')$ such that $f(x) - f(x') = f'(c)(x - x')$. Taking absolute values, this implies

$$|f(x) - f(x')| = |f'(c)||x - x'| \leq B|x - x'|.$$

   Given any $\varepsilon > 0$, let $\delta = \varepsilon/B$. Then $|x-x'| < \varepsilon/B$ implies $|f(x)-f(x')| < B\cdot\varepsilon/B = \varepsilon$. This shows uniform continuity of f on (a, b).

C) (10) If $A \subset \mathbf{R}$ is uncountably infinite, then A contains some nonempty interval (a, b).

   Solution: This is FALSE. The Cantor set C is a counterexample. The "quick and dirty" approach for giving a reason is to recall that C is obtained by removing a collection of open intervals with total length equal to 1 from the interval $[0, 1]$. Hence what is left cannot contain a nonempty open interval (a, b). If so, then C would contain an interval of length $b - a > 0$. But that cannot be true since it would say C and its complement C' could not fit together inside $[0, 1]$.

   Here is a sketch of a better proof as well. Recall that we showed the Cantor set was uncountably infinite in class using the base-3 description of the elements of the Cantor set – they are all the numbers in $[0, 1]$ that have base-3 expansions using only the digits $0, 2$. So the elements of the Cantor set are in correspondence with binary expansions of numbers in $[0, 1]$, which we know is uncountably infinite by the Cantor diagonal argument. On the other hand, C cannot contain any interval. To see this, note that if $c \in C$ and $\varepsilon > 0$, there will always be elements of the complement of the Cantor set in $(c-\varepsilon, c+\varepsilon)$ – just go "far enough out" in the base three expansion

$$c = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

and change some of the digits. Namely, let n_0 be sufficiently large that that the sum

$$\sum_{n=n_0}^{\infty} \frac{2}{3^n} = \frac{1}{3^{n_0-1}} < \varepsilon.$$

If we change a base-3 digit a_n in the expansion of c for some $n \geq n_0$ from a 0 or 2 to a 1, keeping at least one $a_n = 2$ to the right of that 1 so that the number will not be one of those in which the 1 can be replaced by an alternate expansion with a repeating 2 tail, we get a new number c'. The resulting number c' will satisfy

$$|c - c'| < \sum_{n=n_0}^{\infty} \frac{2}{3^n} < \varepsilon.$$

But it will be in the complement of C because of the digit equal to 1.

D) Let f be continuous on $[a, b]$ and assume $f(x) > 0$ for all $x \in [a, b]$. Then there exists a constant $k > 0$ such that $f(x) \geq k > 0$ for all $x \in [a, b]$.

Solution: This is TRUE. By the Extreme Value Theorem, if $k = \text{glb}\{f(x) \mid x \in [a, b]\}$, then there exists a $c \in [a, b]$ such that $f(c) = k$. But then by assumption $k > 0$ and $f(x) \geq k$ for all $x \in [a, b]$.