MATH 242 – Principles of Analysis
Solutions for Exam 2 – April 1, 2011

I.

A) (20) State and prove the Monotone Convergence Theorem for sequences. (You may give the proof in the case that the sequence is monotone increasing.)

Solution: The Theorem says that every bounded monotone sequence in $\mathbf{R}$ converges to some real number a. If $\{x_n\}$ is monotone increasing, let $a = \text{lub}(\{x_n \mid n \geq 1\})$. then for all $\varepsilon > 0$, $a - \varepsilon$ is not an upper bound for $\{x_n \mid n \geq 1\}$. Hence there exists some n_0 such that $a - \varepsilon < x_{n_0} \leq a$. Since the sequence is monotone increasing, for all $n \geq n_0$, we have $a - \varepsilon < x_{n_0} \leq x_n \leq a$. But this implies that $|x_n - a| < \varepsilon$ for all $n \geq n_0$. Hence $x_n \to a$.

B) (10) Suppose that x_n is a sequence of strictly *negative* numbers and $x_{n+1}/x_n \leq 1$ for all $n \geq 1$. Show that $\lim_{n\to\infty} x_n = a$ exists in $\mathbf{R}$ and satisfies $a \leq 0$.

Solution: Since the $x_n < 0$, from the inequality $x_{n+1}/x_n \leq 1$, we get $x_{n+1} \geq x_n$ for all $n \geq 1$. (The inequality reverses when we multiply by the negative number x_n.) This implies the sequence is *monotone increasing*. Since it is also bounded above by 0, part (A) shows that $x_n \to a$ for some $a \leq 0$.

II. (20) Suppose $\{x_n\}$ is a sequence such that $|x_n - 10| < 100$ for all $n \geq 1$. Show that there exists some number $a \in [-90, 110]$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to a$. State any "big theorems" you are using.

Solution: Since $|x_n - 10| < 100$ for all n, it follows that $-100 < x_n - 10 < 100$, so $-90 < x_n < 110$. Hence $|x_n| \leq 110$ for all n, so this is a bounded sequence. By the Bolzano-Weierstrass theorem, there must be a convergent subsequence $x_{n_k} \to a$ for some $a \in \mathbf{R}$. The limit must also lie in the interval $[-90, 110]$ because of the "order limit theorem" (Corollary 2.2.8 in our text). This result says: If $a \leq x_n \leq b$ for all n, and $x_n \to c$, then $a \leq c \leq b$ also.

III. (20) Show using the ε, δ definition that $\lim_{x\to 2} x^2 - x = 2$.

Solution: ("prep": $|x^2 - x - 2| = |x - 2||x + 1|$. If $|x - 2| < 1$, then $1 < x < 3$, so $|x + 1|$ satisfies $2 < |x + 1| < 4$.)

proof: Given $\varepsilon > 0$, let $\delta = \min\left(1, \frac{\varepsilon}{4}\right)$. Then for all x with $0 < |x - 2| < \delta$, $1 < x < 3$, so $|x + 1| < 4$. Therefore, for all such x,

$$|x^2 - x - 2| = |x - 2||x + 1| < \frac{\varepsilon}{4} \cdot 4 = \varepsilon.$$

This shows $\lim_{x\to 2} x^2 - x = 2$.//

IV. True/False. For each true statement, give a short proof. For each false statement, give a counterexample or reason.

A) (10) If $\{x_n\}$ diverges, then for all $a \in \mathbf{R}$, there exist $\varepsilon_0 > 0$ and $n_0 \in \mathbf{N}$ such that $|x_n - a| \geq \varepsilon_0$ for all $n \geq n_0$.

Solution 1: This is FALSE. A counterexample is the sequence $x_n = (-1)^n$. If $a = 1$ (or $a = -1$), note there is no n_0 and no $\varepsilon_0 > 0$ such that $|x_n - a| \geq \varepsilon_0$ for *all* $n \geq n_0$ because $|(-1)^n - 1| = 0$ for all even n. (Similarly, $|(-1)^n + 1| = 0$ for all odd n.)

Solution 2: We can also see that this is FALSE if we think of the meaning of the statement: For all $a \in \mathbf{R}$, $x_n \not\to a$ (that is, x_n *does not converge to* a.). In symbolic form, by negating the definition of the statement $x_n \to a$, this says

$$\forall a, \exists \varepsilon_0 > 0, \forall n_0, \exists n \geq n_0, \;\; s.t. \; |x_n - a| \geq \varepsilon_0.$$

(Note: the quantifier on the n is $\exists$, not $\forall$. The given statement would certainly imply that the sequence does not converge to a, but it is not necessarily true for divergent sequences, as the example in the first solution shows.)

B) (10) It is possible to find a function $f : \mathbf{R} \to \mathbf{R}$ that is continuous at $c = \sqrt{2}$, with $f(\sqrt{2}) = 0$, but satisfying $f(x) = 1$ for all rational x.

Solution 1: This is FALSE. If $\{x_n\}$ is any sequence of rational numbers with $x_n \to \sqrt{2}$, then by a standard property of continuous functions, $f(x_n) \to f(\sqrt{2})$. But $f(x_n) = 1$ for all n, and $f(\sqrt{2}) = 0$. So this is not possible.

Solution 2: We can also see that this is FALSE if we apply the definition of continuity directly. Aiming for a contradiction, suppose there were such a function. Then for all $\varepsilon > 0$, there would exist $\delta > 0$ such that $|f(x) - f(\sqrt{2})| < \varepsilon$ for all x with $|x - \sqrt{2}| < \delta$. If $f(\sqrt{2}) = 0$ and we take $\varepsilon < 1$, then the interval defined by $|y - 0| < \varepsilon$ does not contain 1. But every interval defined by $|x - \sqrt{2}| < \delta$ for $\delta > 0$ contains rational numbers x and for those x, $f(x) = 1$. This is a contradiction which shows no such f exist.

C) (10) The sequence defined by $x_1 = 1$ and $x_n = \sqrt{3x_{n-1} + 1}$ for $n \geq 2$ is monotone increasing.

Solution: This is TRUE. We have $x_1 = 1$ and $x_2 = \sqrt{4} = 2$. so $x_2 > x_1$. This is the base case for an induction. Assume $x_{k+1} \geq x_k$ for a given k. Then by the induction hypothesis,

$$x_{k+2} = \sqrt{3x_{k+1} + 1} \geq \sqrt{3x_k + 1} = x_{k+1}.$$

Therefore $\{x_n\}$ is increasing by induction.

Extra Credit. (10) Assume that $\{x_n\}$ is a sequence that converges to a. Construct a new sequence $\{y_n\}$ by making y_n the average of the first n terms in $\{x_n\}$: $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$. Show that y_n also converges to a.

Solution: Here's the idea. By the triangle inequality, note that

$$
\begin{aligned}
|y_n - a| &= \left| \frac{x_1 + x_2 + \cdots + x_n}{n} - a \right| \\
&= \frac{1}{n} |x_1 + x_2 + \cdots + x_n - na| \\
&\leq \frac{1}{n} \left(|x_1 - a| + \cdots + |x_n - a| \right).
\end{aligned}
$$

Since $\{x_n\}$ converges to a, recall that we know the sequence is bounded. This implies that there exists M such that $|x_n - a| \leq M$ for all n as well. Now, given $\varepsilon > 0$, there exists N such that $|x_n - a| < \varepsilon/2$ for all $n \geq N$. So if $n \geq N$, continuing from the last line above,

$$
\begin{aligned}
&\leq \frac{1}{n}(|x_1 - a| + \cdots + |x_{N-1} - a|) + \frac{n - N + 1}{n} \frac{\varepsilon}{2} \\
&< \frac{NM}{n} + \frac{\varepsilon}{2}.
\end{aligned}
$$

Since we can think of the N that works here as fixed, but n is still allowed to grow, note that we can now find n large enough so that $\frac{NM}{n} < \frac{\varepsilon}{2}$ as well by taking $n > \frac{2NM}{\varepsilon}$. In other words, given $\varepsilon > 0$, we take $n_0 = \max(N, \frac{2NM}{\varepsilon})$. Then $n \geq n_0$ implies $|y_n - a| < \varepsilon$.