Mathematics 242 – Principles of Analysis
Solutions for Exam 1 – February 25, 2011

I.

A) State the Axiom of Completeness for the real number system.

Solution: Every nonempty set of real numbers that is bounded above has a least upper bound. (This is an *axiom* – we take it as an unproved assertion describing properties of the real number system.)

B) Prove that if A and B are bounded sets of real numbers with $\text{lub}(A) < \text{lub}(B)$, then there exists a single $y \in B$ satisfying $y > x$ for all $x \in A$.

Solution: Since $\text{lub}(A) < \text{lub}(B)$, $\text{lub}(A)$ is not an upper bound for B. Hence there exists at least one $y_0 \in B$ such that $\text{lub}(A) < y_0 \leq \text{lub}(B)$. (The subscript is one way mathematicians often indicate that they are referring to one particular element of a set.) But then, by definition, every $x \in A$ satisfies $x \leq \text{lub}(A)$. So $x < y_0$ for all $x \in A$.

C) Let $A = \cap_{n=1}^{\infty} \left(\frac{-1}{2n}, 1 + \frac{1}{2n}\right)$. Explain why A is bounded below and determine $\text{glb}(A)$.

Solution: It is certainly true that $\frac{-1}{2}$ is a lower bound for A. But in fact, given any $\varepsilon > 0$, there exist $n \in \mathbf{N}$ such that $\frac{1}{2n} < \varepsilon$, and hence $\frac{-1}{2n} > -\varepsilon$. We can see that $A = [0, 1]$, so $\text{glb}(A) = 0$.

II.

A) Prove that every interval (a, b) with $0 < a < b$ contains a rational number $\frac{m}{n}$.

Solution: Since $\mathbf{N}$ is not bounded, there exists $n \in \mathbf{N}$ such that $n > \frac{1}{b-a}$, or equivalently $\frac{1}{n} < b - a$. Pick any one such n and fix it. For that fixed n, let $M = \{m \in \mathbf{N} \mid m > na\}$. This set is not empty (again since $\mathbf{N}$ is not bounded). So by the Well Ordering Property, it contains a smallest element. Call that m_0. Now $m_0 > na$, so $\frac{m_0}{n} > a$. On the other hand, $m_0 - 1 \neq M$, so $m_0 - 1 \leq na$. This implies $\frac{m_0}{n} \leq a + \frac{1}{n} < a + b - a = b$. Combining these last facts shows that the rational number $\frac{m_0}{n} \in (a, b)$.

B) Use part A to show that every interval (a, b) with $a < b < 0$ contains a rational number $\frac{m}{n}$.

Solution: If $a < b < 0$, then $0 < (-b) < (-a)$. By part A, there is a rational number $\frac{-m}{n}$ satisfying $(-b) < \frac{-m}{n} < (-a)$. But then, multiplying these inequalities by -1, $a < \frac{m}{n} < b$. Since $\frac{m}{n}$ is also rational, we are done.

III. Let $f : \mathbf{N} \to \mathbf{R}$ be defined by $f(1) = 2$ and $f(n + 1) = \frac{2f(n)-1}{3}$ for all $n \geq 1$. Using mathematical induction, show that $f(n) > -1$ for all $n \geq 1$.

Solution: The base case is $n = 1$. By definition, $f(1) = 2 > -1$. Now assume as the induction hypothesis that $f(k) > -1$, and consider $f(k+1) = \frac{2f(k)-1}{3}$. Then the properties of order imply

$$\begin{aligned} 2f(k) &> -2 \\ \text{hence } 2f(k) - 1 &> -3 \\ \text{hence } f(k+1) = \frac{2f(k)-1}{3} &> -1, \end{aligned}$$

which is what we wanted to show. By the principle of mathematical induction, $f(n) > -1$ for all $n \geq 1$.

IV. True-False. For each true statement give a short proof or reason; for each false statement give a counterexample.

A) If A and B are sets of real numbers with the property that $a > b$ for all $a \in A$ and all $b \in B$, then $\text{glb}(A) > \text{lub}(B)$.

Solution: This is FALSE. A counterexample – let $B = (-\infty, 0)$ and $A = (0, \infty)$ in $\mathbf{R}$. Then $a > b$ for all $a \in A$ and all $b \in B$. But $\text{glb}(A) = 0 = \text{lub}(B)$.

B) (10) For all $n \geq 1$, $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k} = \frac{3^n}{2^n}$.

Solution: This is TRUE. By the binomial theorem,

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

for all real numbers x, y. We get the desired equality by making $x = \frac{1}{2}$, $y = 1$.

C) (10) The smallest c such that $\{x \in \mathbf{R} \mid |x+1| + |x+10| = c\}$ is not empty is $c = 11$.

Solution: This is FALSE. $|x+1| + |x+10|$ represents the sum of the distance from x to -1 and the distance from x to -10 along the number line. For any x with $-10 \leq x \leq -1$, the sum of these distances will be $9 < 11$.

Extra Credit (10) *Caution: this problem may be "habit forming." Only attempt after finishing the rest of the exam!* Can you find a 1-1, onto function $f : \mathbf{R} \to (-1, 1]$? If so, describe one by giving a formula or a graph. If there is no such function, prove it.

Solution: There is such a function. We can construct one as follows: First take any strictly increasing onto function $f : \mathbf{R} \to (-1, 1)$ such as $f(x) = \frac{2}{\pi} \arctan(x)$. This "misses" 1, of course, so we have to modify f to get 1 in the range. If we define a new function g with $g(1) = 1$, but $g(x) = f(x)$ for all $x \neq 1$, then we have missed $f(1) = \frac{1}{2}$ in the range of f.

But we can make $g(2) = f(1) = \frac{1}{2}$, then $g(3) = f(2)$, etc. to get a function that is still 1-1 and onto:

$$\begin{cases} g(x) = f(x) = \frac{2}{\pi}\arctan(x) & \text{if } x \notin \mathbf{N} \\ g(1) = 1 & \\ g(n) = f(n-1) & \text{if } n \geq 2 \in \mathbf{N}. \end{cases}.$$

(This is a mathematical version of the "Hilbert Hotel" with rooms numbered by $n \in \mathbf{N}$ – there's always room for one more guest if we shift everyone else down one room!)